# A General Trotter-Kato Formula for a Class of Evolution Operators with Examples

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**Abstract**

In this article we prove new results concerning the existence and various properties of an evolution system $U_{A+B}(t,s)_{0\leq s\leq t\leq T}$ generated by the sum $-(A(t)+B(t))$ of two linear, time-dependent and generally unbounded operators defined on time-dependent domains in a complex and separable Banach space $\mathcal{B}$. In particular, writing $\mathcal{L}(\mathcal{B})$ for the algebra of all linear bounded operators on $\mathcal{B}$, we can express $U_{A+B}(t,s)_{0\leq s\leq t\leq T}$ as the strong limit in $\mathcal{L}(\mathcal{B})$ of a product of the holomorphic contraction semigroups generated by $-A(t)$ and $-B(t)$, respectively, thereby proving a product formula of the Trotter-Kato type under very general conditions which allow the domain $\mathcal{D}(A(t)+B(t))$ to evolve with time provided there exists a fixed set $\mathcal{D}\subset\cap_{t\in[0,T]}\mathcal{D}(A(t)+B(t))$ everywhere dense in $\mathcal{B}$. We obtain a special case of our formula when $B(t)=0$, which, in effect, allows us to reconstruct $U_A(t,s)_{0\leq s\leq t\leq T}$ very simply in terms of the semigroup generated by $-A(t)$. We then illustrate our results by considering various examples of non-autonomous parabolic initial-boundary value problems, including one related to the theory of time-dependent singular perturbations of self-adjoint operators. We finally mention what we think remains an open problem for the corresponding equations of Schrödinger type in quantum mechanics.

## 1 Introduction and Outline

It is well-known that the Hille-Yosida theory of semigroups and its numerous extensions regarding the construction of evolution operators on Banach spaces has had and still has far reaching applications to the analysis of certain linear or nonlinear, deterministic or stochastic, partial differential equations with time-independent or time-dependent coefficients. In many instances that may encompass parabolic equations, hyperbolic equations or Schrödinger equations, to name only a few, it is indeed possible to reformulate a given initial and boundary-value problem as one related to evolution equations on suitably chosen functional spaces. The mathematical investigation of such a problem concerning for example the existence and the uniqueness of various types of solutions, the relations among them, their various representations and their asymptotic behavior for large times, then becomes intimately related to the properties of the corresponding linear propagator (see, for instance, [22], [29], [35] and [38] for general references regarding the deterministic case as well as [9] for the stochastic case). Among those properties, perturbation formulae of the Trotter-Kato type such as those stated in [6], [7], [14], [26], [27] or [36] for holomorphic or more general semi-groups are of particular importance for the understanding of certain basic questions in applied mathematics or mathematical physics that can be formulated in terms of *autonomous* partial differential equations; thus, a strongly convergent product formula of the form

$$\exp\left[-t(A+B)\right] = \lim_{n\to+\infty}\left(\exp\left[-\frac{t}{n}A\right]\exp\left[-\frac{t}{n}B\right]\right)^{n} \tag{1}$$

with $t \in \mathbb{R}^{+}$ and $A$, $B$ *time-independent* linear operators on a Banach space satisfying certain conditions, allows one to relate the solutions of certain evolution problems to the theory of Wiener integrals through the celebrated Feynman-Kac formula (see, for instance, [31]). On the other hand, in the realm of quantum mechanics a slightly modified version of (1) also allows a rigorous construction of the so-called Feynman path integral representation of the solutions to Schrödinger equations with *time-independent* potentials (see, for instance, [4], [19] and [27]). Consequently, a question that arises naturally is whether formulae of the form (1) can be generalized to the case where the linear operators $A(t)$ and $B(t)$ depend explicitly on the time variable in some way; it turns out that such a generalization was indeed carried out in [13] when both $A(t)$ and $B(t)$ are the infinitesimal generators of $\mathcal{C}_0$-contraction semigroups on a Banach space for every $t$, under the additional restriction that the domain $\mathcal{D}(A(t)+B(t))$ of the operator sum $A(t)+B(t)$ be *time-independent*; this was nonetheless sufficient to enable the author of [13] to give a precise mathematical meaning to the Feynman path integral representation in the case of Schrödinger equations with certain *time-dependent* potentials. With further hypotheses regarding the continuity properties of $A(t)$ and $B(t)$ as functions of $t$, a generalization of (1) was also obtained in [18] where the authors were able to prove the convergence of their approximations in the operator norm-topology rather than just in the strong topology.

There are, however, a host of important situations where $\mathcal{D}(A(t)+B(t))$ does depend explicitly on time, thereby making some of the arguments of [13] and [18] inapplicable; as a concrete class of examples which will motivate some of the hypotheses of the theory we develop below, let $D \subset \mathbb{R}^{d}$ be an open bounded domain with a smooth boundary $\partial D$ (see, for instance, [2] for a definition of this and related concepts); let $T \in \mathbb{R}^{+}_{*} := \mathbb{R}^{+}\diagdown\{0\}$ and let us consider parabolic initial-boundary value problems of the form

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \operatorname{div}(k(x,t)\nabla u(x,t)) - \varkappa u(x,t), \quad (x,t) \in D \times (0,T]\,, \\
u(x,0) &= u_0(x), \quad x \in D, \\
\frac{\partial u(x,t)}{\partial n(k)} &= 0, \quad (x,t) \in \partial D \times (0,T]\,,
\end{aligned} \tag{2}
$$

with $\varkappa \in \mathbb{R}^+$ a parameter and where the last relation in (2) stands for the conormal derivative of $u$ relative to the matrix-valued function $k$. We assume that the following hypotheses hold (here and below we use the standard notations for the usual spaces of Lebesgue integrable functions and for the corresponding Sobolev spaces on regions of Euclidean space; we also write $c$ for all the irrelevant constants that occur in the various estimates unless we specify these constants otherwise):

(K) The function $k : D \times [0,T] \mapsto \mathbb{R}^{d^2}$ is matrix-valued and for every $i,j \in \{1,...,d\}$ we have $k_{i,j} = k_{j,i} \in L^\infty(D \times (0,T), \mathbb{R})$; moreover, there exists a constant $\underline{k} \in \mathbb{R}^+_*$ such that the inequality

$$
(k(x,t)q,q)_{\mathbb{R}^d} \geq \underline{k}\,|q|^2 \tag{3}
$$

holds uniformly in $(x,t) \in D \times [0,T]$ for all $q \in \mathbb{R}^d$, where $(.,.)_{\mathbb{R}^d}$ and $|.|$ denote the Euclidean inner product and the induced norm in $\mathbb{R}^d$, respectively; finally, there exist constants $c_* \in \mathbb{R}^+_*$, $\sigma \in \left(\frac{1}{2}, 1\right]$, such that the Hölder continuity estimate

$$
\max_{i,j\in\{1,\dots,d\}} |k_{i,j}(x,t) - k_{i,j}(x,s)| \leq c_* \,|t-s|^\sigma
$$

is valid for every $x \in D$ and every $s,t \in [0,T]$.

(I) The initial datum satisfies $u_0 \in L^2(D, \mathbb{R})$.

As is well-known, Hypothesis (K) allows one to construct a self-adjoint, positive realization of the elliptic partial differential operator with conormal boundary conditions in (2). In fact, let us write $(.,.)_2$ and $\|.\|_2$ for the inner product and the induced norm in $L^2(D, \mathbb{C})$, respectively, together with $(.,.)_{1,2}$ and $\|.\|_{1,2}$ for the inner product and the induced norm in $H^1(D, \mathbb{C})$, respectively; let $(.,.)_{\mathbb{C}^d}$ be the standard inner product in $\mathbb{C}^d$. Then, for the Hermitian sesquilinear form $\mathsf{a}$:$[0,T] \times H^1(D,\mathbb{C}) \times H^1(D,\mathbb{C}) \mapsto \mathbb{C}$ defined by

$$
\mathsf{a}(t,v,w) := \int_D dx\,(k(x,t)\nabla v(x), \nabla w(x))_{\mathbb{C}^d} \tag{4}
$$

we have the estimates

$$
\begin{aligned}
|\mathsf{a}(t,v,w)| &\leq c\,\|v\|_{1,2}\,\|w\|_{1,2}\,, \\
\mathsf{a}(t,v,v) &\geq \underline{k}\left(\|v\|_{1,2}^2 - \|v\|_2^2\right) \geq 0
\end{aligned} \tag{5}
$$

uniformly in $t \in [0,T]$ for every $v, w \in H^1(D, \mathbb{C})$, as well as

$$|\mathsf{a}(t,v,w) - \mathsf{a}(s,v,w)| \leq c\,|t-s|^{\sigma}\,\|v\|_{1,2}\,\|w\|_{1,2} \tag{6}$$

for every $s, t \in [0,T]$; consequently, the operator

$$A(t) := -\operatorname{div}(k(.,t)\nabla) + \varkappa \tag{7}$$

is indeed self-adjoint and positive in $L^2(D, \mathbb{C})$ on the *time-dependent* domain given by

$$\mathcal{D}(A(t)) = \left\{ v \in H^1(D,\mathbb{C}) : A(t)v \in L^2(D,\mathbb{C}), ((A(t)-\varkappa)v, w)_2 = \mathsf{a}(t,v,w) \right\} \tag{8}$$

where the last relation in (8) holds for every $w \in H^1(D,\mathbb{C})$. Then for any $t \in [0,T]$, $-A(t)$ is the infinitesimal generator of a holomorphic semigroup of contractions $\exp\left[-sA(t)\right]_{s \geq 0}$ in $L^2(D,\mathbb{C})$, and also generates there an evolution system $U_A(t,s)_{0 \leq s \leq t \leq T}$ given by

$$U_A(t,s)v = \begin{cases} v & \text{if } t = s, \\ \int_D dy G_A(.,t;y,s)v(y) & \text{if } t > s, \end{cases} \tag{9}$$

whose range satisfies

$$\operatorname{Ran} U_A(t,s) \subseteq \mathcal{D}(A(t))$$

for every $s, t$ with $0 \leq s < t \leq T$. Here we denote by $G_A$ the parabolic Green's function associated with (2) (see, for instance, [24], [25], [29] or [35] for other typical constructions of this kind). This means that it becomes possible to investigate the existence and the various properties of solutions to (2) through the integral relation

$$u(.,t) = \int_D dy G_A(.,t;y,0)u_0(y)$$

in $L^2(D,\mathbb{R})$.

Let us now perturb the partial differential operator in (2) by considering initial-boundary value problems of the form

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = \operatorname{div}(k(x,t)\nabla u(x,t)) - (l(x,t), \nabla u(x,t))_{\mathbb{R}^d} - (\varkappa + \varepsilon m(x,t))u(x,t), \\
&\quad (x,t) \in D \times (0,T], \\
&\quad u(x,0) = u_0(x), \quad x \in D, \\
&\frac{\partial u(x,t)}{\partial n(k)} = 0, \quad (x,t) \in \partial D \times (0,T],
\end{aligned} \tag{10}$$

with $\varepsilon \in \mathbb{R}^+$ a parameter and with the following additional hypotheses regarding the lower-order differential operators, where we assume without restricting the generality that the constants $c_*$ and $\sigma$ are the same as in Hypothesis (K):

(L) Each component of the vector-field $l : D \times [0,T] \mapsto \mathbb{R}^d$ satisfies $l_i \in L^\infty(D \times (0,T), \mathbb{R})$ and the Hölder continuity estimate

$$\max_{i \in \{1,\dots,d\}} |l_i(x,t) - l_i(x,s)| \le c_* |t-s|^\sigma$$

holds for every $x \in D$ and every $s,t \in [0,T]$.

(M) We have $m \in L^\infty(D \times (0,T), \mathbb{R}^+)$ along with

$$|m(x,t) - m(x,s)| \le c_* |t-s|^\sigma$$

for every $x \in D$ and every $s,t \in [0,T]$.

As is the case for (7), it is also possible to construct a realization of the partial differential operator

$$\begin{aligned} C_\varepsilon(t) &:= -\operatorname{div}(k(.,t)\nabla) + \varkappa + (l(.,t), \nabla)_{\mathbb{R}^d} + \varepsilon m(.,t) \\ &:= A(t) + B_\varepsilon(t) \end{aligned} \tag{11}$$

in (10) by considering the sesquilinear form $\mathsf{c}_\varepsilon : [0,T] \times H^1(D,\mathbb{C}) \times H^1(D,\mathbb{C}) \mapsto \mathbb{C}$ defined by

$$\mathsf{c}_\varepsilon(t,v,w) := \mathsf{a}(t,v,w) + \varkappa(v,w)_2 + \mathsf{b}_\varepsilon(t,v,w) \tag{12}$$

with $\mathsf{a}(t,v,w)$ given by (4) and

$$\mathsf{b}_\varepsilon(t,v,w) := \int_D dx\, (l(x,t), \nabla v(x))_{\mathbb{C}^d}\, \overline{w}(x) + \varepsilon \int_D dx m(x,t) v(x) \overline{w}(x). \tag{13}$$

In fact, thanks to Hypotheses (L), (M) and by elementary arguments we get the estimates

$$\begin{aligned} |\mathsf{b}_\varepsilon(t,v,w)| &\le c \|v\|_{1,2} \|w\|_2 , \\ \operatorname{Re} \mathsf{b}_\varepsilon(t,v,v) &\ge -\frac{k}{2} \|v\|_{1,2}^2 - c \|v\|_2^2 \end{aligned} \tag{14}$$

uniformly in $t \in [0,T]$ for every $v, w \in H^1(D,\mathbb{C})$, as well as

$$|\mathsf{b}_\varepsilon(t,v,w) - \mathsf{b}_\varepsilon(s,v,w)| \le c |t-s|^\sigma \|v\|_{1,2} \|w\|_2 \tag{15}$$

for every $s,t \in [0,T]$, with the norm $\|w\|_2$ rather than $\|w\|_{1,2}$ in (14) and (15). Consequently, this leads to the realization of the lower-order operator $B_\varepsilon(t)$ in $L^2(D,\mathbb{C})$ on the *time-independent* domain

$$\mathcal{D}(B_\varepsilon(t)) = H^1(D,\mathbb{C})$$

with

$$(B_\varepsilon(t)v, w)_2 = \mathsf{b}_\varepsilon(t,v,w)$$

and

$$\|B_\varepsilon(t)v\|_2 \le c \|v\|_{1,2} \tag{16}$$

for every $t \in [0,T]$, any $v \in H^1(D,\mathbb{C})$ and each $w \in L^2(D,\mathbb{C})$, and thereby to the realization of (11) as an operator in $L^2(D,\mathbb{C})$ on the *time-dependent* domain

$$\mathcal{D}(C_\varepsilon(t)) = \mathcal{D}(A(t)) \cap H^1(D,\mathbb{C}) = \mathcal{D}(A(t))$$

for every $t \in [0,T]$. Moreover, as is the case for $-A(t)$ the operator $-C_\varepsilon(t)$ also generates a holomorphic semigroup and an evolution system $U_{A+B_\varepsilon}(t,s)_{0\leq s\leq t\leq T}$ given by

$$U_{A+B_\varepsilon}(t,s)v = \begin{cases} v \quad \text{if} \quad t=s, \\ \int_D dy G_{A+B_\varepsilon}(.,t;y,s)v(y) \quad \text{if} \quad t>s \end{cases} \tag{17}$$

in $L^2(D,\mathbb{C})$, whose range satisfies

$$\operatorname{Ran} U_{A+B_\varepsilon}(t,s) \subseteq \mathcal{D}(A(t))$$

for every $s,t$ with $0 \leq s < t \leq T$ and where $G_{A+B_\varepsilon}$ is the parabolic Green's function associated with the differential operator in (10). These two assertions follow from the general theory developed in [35] since we can infer successively from (5), (6), (14) and (15) that the estimates

$$|\mathsf{c}_\varepsilon(t,v,w)| \leq c \left\|v\right\|_{1,2} \left\|w\right\|_{1,2}, \tag{18}$$

$$\operatorname{Re}\mathsf{c}_\varepsilon(t,v,v) \geq \frac{\underline{k}}{2}\left\|v\right\|_{1,2}^2 - (\underline{k}+c)\left\|v\right\|_2^2 \tag{19}$$

hold uniformly in $t \in [0,T]$ for every $v,w \in H^1(D,\mathbb{C})$, and that

$$|\mathsf{c}_\varepsilon(t,v,w) - \mathsf{c}_\varepsilon(s,v,w)| \leq c\,|t-s|^\sigma \left\|v\right\|_{1,2}\left\|w\right\|_{1,2} \tag{20}$$

holds for every $s,t \in [0,T]$.

In the realm of this class of examples the natural questions we want to ask are whether we can reconstruct the evolution system $U_A(t,s)_{0\leq s\leq t\leq T}$ in terms of the contraction semigroup $\exp\left[-sA(t)\right]_{s\geq 0}$ in a simple manner, and more generally whether we can express (17) in terms of the unperturbed system (9) through some kind of generalization of (1). Even the first question is not trivial, as the various relations known thus far between $U_A(t,s)_{0\leq s\leq t\leq T}$ and $\exp\left[-sA(t)\right]_{s\geq 0}$ are notoriously complicated ones (see, for instance, [29] and [35]).

In order to motivate further the theory we develop below, it is worth noting here that under the above hypotheses the operator $B_\varepsilon(t)$ is always a relatively bounded perturbation of the operator $A(t)$ in the sense of [22]. In fact, aside from the inclusion

$$\mathcal{D}(A(t)) \subseteq \mathcal{D}(B_\varepsilon(t))$$

we also have

$$\left\|v\right\|_{1,2}^2 \leq \underline{k}^{-1}\left((A(t)-\varkappa)v,v\right)_2 + \left\|v\right\|_2^2 \leq \underline{k}^{-1}\left\|(A(t)-\varkappa)v\right\|_2\left\|v\right\|_2 + \left\|v\right\|_2^2$$

as a consequence of (5), (8) and Schwarz inequality, which implies

$$\left\|v\right\|_{1,2} \leq c\left(\left\|A(t)v\right\|_2 + \left\|v\right\|_2\right)$$

for every $v \in \mathcal{D}(A(t))$ since $\|v\|_{1,2}^{-1} \|v\|_2 \leq 1$ when $v \neq 0$. Consequently, from the last relation and (16) we obtain

$$\|B_\varepsilon(t)v\|_2 \leq c\left(\|A(t)v\|_2 + \|v\|_2\right)$$

for every $t \in [0,T]$ and any $v \in \mathcal{D}(A(t))$, which is the desired assertion.

As we shall see in Section 4, similar questions can be raised for other classes of concrete examples, a case in point being the class of time-dependent singular perturbations of self-adjoint differential operators which are supported on a finite or discrete set of points in Euclidean space (see, for instance, [3], [8], [11], [12] and [15] for general references concerning such problems).

Although (10) is inherently variational, it is equally plain that it is formally a particular example of an abstract evolution problem of the form

$$\begin{aligned} \frac{du(t)}{dt} &= -(A(t)+B(t))u(t), \quad t \in (s,T]\,, \\ u(s) &= u_s \end{aligned} \tag{21}$$

defined in a complex and separable Banach space $\mathcal{B}$. In the sequel we shall investigate (21) from the point of view we just outlined under appropriate hypotheses concerning $A(t)$ and $B(t)$ when $\mathcal{D}(A(t)+B(t))$ may be *time-dependent*, but without reference to any kind of variational structure in the abstract setting. Accordingly, we shall organize the remaining part of this article in the following way: in Section 2 we state and discuss our main theorem regarding the existence of an evolution system $U_{A+B}(t,s)_{0\leq s\leq t\leq T}$ concerning (21) and a related extension of (1), for a suitable class of $A(t)$'s and of time-dependent perturbations $B(t)_{0\leq t\leq T}$. There we also put our result into a broader perspective by comparing our way of constructing the $U_{A+B}(t,s)$'s with other known methods such as those put forward in [1], [30] or in the review article [33]. We prove our main result in Section 3; our general framework in that section is the theory of evolution operators as developed in [35], which indeed motivated our choice of the $A(t)$'s and the $B(t)$'s in the first place. We illustrate our main statements by means of several examples in Section 4, aside from also considering there examples showing that some of our hypotheses, *albeit* natural, sufficient and indeed verifiable in a host of important situations, are not necessary for our product formula to hold. In this context it is worth pointing out that there are two well-known analytical tools which play an important rôle in our analysis of some of those examples, namely, Euler's summation formula and Krein's formula for resolvents (see, for instance, [17] and [3], respectively). Finally, we refer the reader to [37] for a short announcement of our result and a very brief sketch of its proof.

## 2 Statement and Discussion of the Main Result

In the sequel we write $\|.\|$ for the norm in $\mathcal{B}$ and $\|.\|_\infty$ for the usual operator-norm in $\mathcal{L}(\mathcal{B})$, the Banach algebra of all bounded linear operators on $\mathcal{B}$ . According to what we outlined in the preceding section, we wish to construct an evolution system $U_{A+B}(t,s)_{0\leq s\leq t\leq T}$ for Problem (21) which we can express in terms of the semigroups generated by $A(t)$ and $B(t)$ through a suitable generalization of (1), without ever requiring that the domains $\mathcal{D}(C^{\delta}(t))$ of the fractional powers of $C(t)$ for $\delta\in(0,1]$ be time-independent, where $C(t):=A(t)+B(t)$. To this end we assume that the following hypotheses are valid (see, for instance, [35] for the basic definitions and properties):

(A1) The linear operator $-A(t)$ is the infinitesimal generator of a holomorphic semigroup $\exp\left[-sA(t)\right]_{s\geq 0}$ on $\mathcal{B}$ for every $t\in[0,T]$ and we have $0\in\rho(A(t))$ for any such $t$, where $\rho(A(t))$ denotes the resolvent set of $A(t)$.

(A2) The function $t\mapsto A^{-1}(t)$ is continuously differentiable with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ and there exist constants $a_2\in\mathbb{R}_*^+$, $\tilde{a}_2\in(0,1]$ such that the Hölder continuity estimate

$$\left\|\frac{dA^{-1}(t)}{dt}-\frac{dA^{-1}(s)}{ds}\right\|_{\infty}\leq a_2\left|t-s\right|^{\tilde{a}_2} \tag{22}$$

is valid for every $s,t\in[0,T]$.

As is well-known, Hypothesis (A1) implies the existence of constants $\theta\in\left(0,\frac{\pi}{2}\right)$, $c_*\in\mathbb{R}_*^+$ such that the inclusion $\mathcal{S}_\theta\subseteq\rho(A(t))$ and the inequality

$$\left\|R(A(t),\lambda)\right\|_{\infty}\leq c_*\left(1+|\lambda|\right)^{-1} \tag{23}$$

hold for every $t\in[0,T]$ and any $\lambda\in\mathcal{S}_\theta$, where

$$R(A(t),\lambda):=\left(A(t)-\lambda\right)^{-1}$$

and

$$\mathcal{S}_\theta:=\left\{\lambda\in\mathbb{C}:|\arg\lambda|\geq\theta\right\}\cup\{0\}. \tag{24}$$

Furthermore, Hypotheses (A1) and (A2) also imply the differentiability of the function $t\mapsto R(A(t),\lambda)$ on $[0,T]$ with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$, whose derivative we require to satisfy the following hypothesis:

(A3) There exist constants $a_3\in\mathbb{R}_*^+$, $\tilde{a}_3\in(0,1]$ such that the inequality

$$\left\|\frac{\partial}{\partial t}R(A(t),\lambda)\right\|_{\infty}\leq a_3\left|\lambda\right|^{-\tilde{a}_3} \tag{25}$$

holds for every $t\in[0,T]$ and every $\lambda\in\mathcal{S}_\theta\diagdown\{0\}$.

Hypotheses (A1)-(A3) are the building blocks of the existence theory of solutions to non-autonomous linear parabolic equations developed in [35] when $\mathcal{D}(A(t))$ varies with time, thereby providing an evolution system $U_A(t,s)_{0\leq s\leq t\leq T}$

for Problem (21) when $B(t) = 0$; however, they are by far not the only sufficient conditions that allow the construction of the $U_A(t,s)$'s, and we shall indeed dwell a bit on this point and on related matters immediately after the statement of our theorem.

Since we have in mind a generalization of (1) to the time-dependent case, it is then natural to ask whether those conditions remain stable under a suitable class of perturbations $B(t)_{0\leq t\leq T}$ of the $A(t)$'s. We shall see that this is indeed the case provided we impose the following hypotheses:

(B1) The linear operator $B(t)$ is closed in $\mathcal{B}$ for every $t \in [0,T]$ and we have $\mathcal{D}(B(t)) \supseteq \mathcal{D}(A(t))$ for any such $t$; moreover, there exist constants $a \in [0,1)$, $b \in \left[0,(1-a)c_*^{-1}\right)$ where $c_*$ is the constant in (23), such that the inequality

$$\|B(t)v\| \leq a\,\|(A(t)-\lambda)v\| + b\,\|v\| \tag{26}$$

holds for every $v \in \mathcal{D}(A(t))$, any $t \in [0,T]$ and each $\lambda \in \mathcal{S}_\theta$.

(B2) The function $t \mapsto B(t)A^{-1}(t)$ is continuously differentiable on $[0,T]$ with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ and there exist constants $b_2 \in \mathbb{R}_*^+$, $\tilde{b}_2 \in (0,1]$ such that the Hölder continuity estimate

$$\left\|\frac{d(B(t)A^{-1}(t))}{dt} - \frac{d(B(s)A^{-1}(s))}{ds}\right\|_\infty \leq b_2\,|t-s|^{\tilde{b}_2}$$

is valid for every $s,t \in [0,T]$.

(B3) The function $t \mapsto B(t)R(A(t),\lambda)$ is continuously differentiable on $[0,T]$ with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ and there exists a constant $c_{**} \in \mathbb{R}_*^+$ such that the inequality

$$\left\|\frac{\partial}{\partial t}(B(t)R(A(t),\lambda))\right\|_\infty \leq c_{**}$$

holds for every $t \in [0,T]$ and each $\lambda \in \mathcal{S}_\theta$.

While Hypothesis (B1) is evidently some kind of relative boundedness condition, we remark that it also imposes a smallness condition on the constant $b$ in (26). This will allow us to prove a crucial ingredient for our upcoming arguments to work, namely, the bounded invertibility of $A(t)+B(t)$ for every $t \in [0,T]$, which means that even in the case of bounded $B(t)$'s the admissible perturbations will be limited to those of small norm. Furthermore, whereas the preceding hypotheses indeed guarantee the existence of the evolution system we alluded to above (see Proposition 1 of Section 3), we note that they are not quite sufficient to allow the generalization of (1) that we want. For this we still impose the following three conditions.

(A4) The semigroup $\exp\left[-sA(t)\right]_{s\geq 0}$ in Hypothesis (A1) is contractive on $\mathcal{B}$ for every $t \in [0,T]$.

(B4) The operator $-B(t)$ is the infinitesimal generator of a holomorphic semigroup of contractions $\exp\left[-sB(t)\right]_{s\geq 0}$ on $\mathcal{B}$ for every $t \in [0,T]$; moreover, the function $t \mapsto R(B(t),\lambda)$ is continuous on $[0,T]$ uniformly in $\lambda \in \mathcal{S}_{\theta^*}$ in the strong topology of $\mathcal{L}(\mathcal{B})$, where $\mathcal{S}_{\theta^*}$ is given by (24) but with

$$\mathcal{S}_{\theta^*} \subseteq \rho(B(t))$$

for some $\theta^* \in \left(0, \frac{\pi}{2}\right)$.

(D) There exists a dense set $\mathcal{D} \subset \mathcal{B}$ satisfying

$$\mathcal{D} \subset \cap_{t\in[0,T]} \mathcal{D}(A(t)+B(t)) \tag{27}$$

such that for every $v \in \mathcal{D}$ we have

$$\sup_{t\in(0,T]} \|A(t)v\| < +\infty \tag{28}$$

and

$$\sup_{t\in(0,T]} \|B(t)v\| < +\infty. \tag{29}$$

As is the case for the operator $A(t)$, Hypothesis (B4) also implies the existence of a constant $c_* \in \mathbb{R}^+_*$ such that the resolvent estimate

$$\|R(B(t),\lambda)\|_\infty \leq c_* \left(1+|\lambda|\right)^{-1} \tag{30}$$

holds for every $t \in [0,T]$ and every $\lambda \in \mathcal{S}_{\theta^*}$; moreover, our arguments below will show that in the particular case of time-independent $B$'s, we can weaken Hypothesis (B4) by only requiring that $\exp\left[-sB\right]_{s\geq 0}$ be a $\mathcal{C}_0$-contraction semigroup.

Under these conditions we can formulate our main result as follows.

**Theorem.** *Assume that Hypotheses (A1)-(A3) and (B1)-(B3) hold. Then there exists an evolution system $U_{A+B}(t,s)_{0\leq s\leq t\leq T}$ solving Problem (21) such that the following properties are valid for all $s,t$ with $0 \leq s < t \leq T$:*
*(1) The range of $U_{A+B}(t,s)$ satisfies*

$$\operatorname{Ran} U_{A+B}(t,s) \subseteq \mathcal{D}(A(t)+B(t)) = \mathcal{D}(A(t)). \tag{31}$$

*Moreover, the operator-valued function $t \mapsto U_{A+B}(t,s)$ is continuously differentiable with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ and we have*

$$\frac{\partial U_{A+B}(t,s)}{\partial t} = -(A(t)+B(t))U_{A+B}(t,s) \in \mathcal{L}(\mathcal{B})$$

*with the estimate*

$$\left\|\frac{\partial U_{A+B}(t,s)}{\partial t}\right\|_\infty \leq c(t-s)^{-1}$$

*for some $c \in \mathbb{R}_*^+$ independent of $s, t$. Finally, the operator-valued function $s \mapsto U_{A+B}(t,s)$ is also differentiable with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ and we have*

$$\frac{\partial U_{A+B}(t,s)}{\partial s} \in \mathcal{L}(\mathcal{B})$$

*with the same estimate as above, namely,*

$$\left\| \frac{\partial U_{A+B}(t,s)}{\partial s} \right\|_\infty \leq c(t-s)^{-1}$$

*where $\frac{\partial U_{A+B}(t,s)}{\partial s}$ stands for the bounded linear extension of $U_{A+B}(t,s)(A(s)+B(s))$ on $\mathcal{B}$.*

*(2) In addition to the above hypotheses, if (A4), (B4) and (D) hold then for all $s, t$ with $0 \leq s \leq t < T$ we have the Trotter-Kato product formula*

$$\begin{aligned} & U_{A+B}(t,s) && (32) \\ = \quad & \lim_{n \to +\infty} \prod_{\gamma=n-1}^{0} \exp\left[-\frac{t-s}{n} A\left(s + \frac{\gamma}{n}(t-s)\right)\right] \exp\left[-\frac{t-s}{n} B\left(s + \frac{\gamma}{n}(t-s)\right)\right] \end{aligned}$$

*in the strong topology of $\mathcal{L}(\mathcal{B})$.*

**Remarks.** (1) Aside from Hypotheses (A1)-(A3), there exist several other sufficient conditions that would have allowed the construction of the $U_A(t,s)$'s when $B(t) = 0$; we refer the reader for instance to [1] for a general and thorough investigation of such conditions and of the relations among them. In particular, we could have used Hypotheses I and II of that paper in the somewhat stronger form introduced in [30] and [33] to get such a result. There also exist various sufficient conditions which could have lead to the existence of perturbed evolution systems $U_{A+B}(t,s)$ for suitable classes of $B(t)$'s, for example those put forward in [30] and [33]. However, a basic difficulty emerges there when one tries to prove a product formula such as (32) for them; thus, while the $U_{A+B}(t,s)$'s of Theorem 9.19 in [33] are only defined for *almost every* $t > s$ and lack differentiability properties, those of Theorem 4.2 in [30] are only *weakly locally differentiable* relative to the time variable and satisfy an equation such as (21) *almost everywhere*. A direct consequence of this is that such evolution systems are not amenable to the method of proof we develop in the next section, which requires the $U_{A+B}(t,s)$'s to be once continuously differentiable in $t$ relative to the strong topology of $\mathcal{L}(\mathcal{B})$; furthermore, such a strong smoothness property does not readily follow from our hypotheses regarding the $B(t)$'s unless we assume more regularity properties on the perturbations (see, for instance, [23] for results in this direction). In short, it is thus far the general framework of [35] that has allowed us to prove the above theorem and to deal with all the examples we have in mind in a relatively simple and direct way . Of course, whether one can prove a Trotter-Kato formula such as (32) under the sole conditions of [30], [33], or under even more general conditions, remains an interesting open problem at this time.

(2) It is clear that the condition $0 \in \rho(A(t)) \cap \rho(B(t))$ stemming from Hypotheses (A1), (B4) is imposed only for convenience, as the conclusions of our theorem still hold without this restriction; in particular, (32) remains unaltered by the addition of constants to $A(t)$ or $B(t)$. It is also clear that if both $A(t)$ and $B(t)$ are independent of $t$, formula (32) reduces to the form (1). However, in the time-dependent case we ought to point out that the first factor on the right-hand side of (32) only involves the contraction semigroup $\exp\left[-sA(t)\right]_{s\geq 0}$ and *not* the full evolution system $U_A(t,s)_{0\leq s\leq t\leq T}$. Then, by choosing $B=0$ in (32) we obtain

$$U_A(t,s) = \lim_{n\to+\infty} \prod_{\gamma=n-1}^{0} \exp\left[-\frac{t-s}{n} A\left(s+\frac{\gamma}{n}(t-s)\right)\right], \tag{33}$$

which provides the new and simple way of reconstructing the $U_A(t,s)$'s from the $\exp\left[-sA(t)\right]$'s we alluded to above. In Section 4 we shall also consider two examples for which we can prove (33) more directly by means of Euler's summation formula.

(3) While (32) and (33) hold in the strong topology of $\mathcal{L}(\mathcal{B})$, an issue of independent interest is whether there might exist simple and natural conditions which would imply the convergence of these approximations in the norm-topology of that space. We refer the reader to [5], [18], [26] and [39] for some results and discussions in this direction in a different context.

(4) Whereas the above conditions are sufficient to ensure the validity of the theorem, they are certainly not optimal since we did not strive for maximal generality. In particular, they are not all necessary when applied to parabolic problems that exhibit a variational structure; this is easy to understand in light of the theory developed in [35] since, in that case, proofs can as a rule be obtained under a weaker set of hypotheses. Typical hypotheses of this kind are, for instance, (4)-(6) and (18)-(20) in the case of (9) and (17), respectively. In particular, it would be highly desirable to get a proof of (32) under hypotheses of that kind, which, in effect, raises the more general question of proving product formulae by means of the theory of time-dependent quadratic forms. To the best of our knowledge this is an open problem, whereas the time-independent case was settled in [21], of which a special case can be found in [10]. We shall come back to this point in Section 4.

(5) Our theorem offers an alternative approach to Kato's theory of non-autonomous parabolic evolution equations which was developed many years ago in [20]. Since that time this theory has been successfully applied to numerous specific problems particularly when the domains of the operators involved are *time-independent* (see, for instance, [19], [20], [29], [35] and the references therein). However, when those domains become *time-dependent* Kato's theory imposes rather strong invariance conditions which are as a rule very difficult to check in practice, particularly in concrete examples of partial differential equations with time-varying boundary conditions such as (10). This remark applies, for instance, to the verification of the first product formula in [28], which, incidentally, does bear some formal resemblance with (33). By contrast, our result

does *not* require any such invariance conditions and thereby allows us to treat a wide class of such models as we shall see below. Finally, we also would like to mention [16] and its numerous references for a systematic account of certain recent probabilistic developments of Kato's theory in the non-autonomous case, including the analysis of the related Feynman-Kac propagators.

We devote the next section of this article to the proof of the above theorem.

# 3 Proof of the Main Result

Our preliminary remark is the following lemma, whose proof is immediate by induction and therefore omitted.

**Lemma 1.** *For every $n \in \mathbb{N}^+ \cap [3, +\infty)$ let $(U_\gamma)_{\gamma \in \{1,...,n\}}$ and $(V_\gamma)_{\gamma \in \{1,...,n\}}$ be two families of operators in $\mathcal{L}(\mathcal{B})$; then the identity*

$$\prod_{\gamma=n}^{1} U_\gamma - \prod_{\gamma=n}^{1} V_\gamma = \prod_{\alpha=n}^{2} U_\alpha \times (U_1 - V_1) \\ + \sum_{\gamma=2}^{n-1} \prod_{\alpha=n}^{\gamma+1} U_\alpha \times (U_\gamma - V_\gamma) \times \prod_{\beta=\gamma-1}^{1} V_\beta + (U_n - V_n) \times \prod_{\beta=n-1}^{1} V_\beta \qquad (34)$$

*holds. Furthermore, for every $n \in \mathbb{N}^+$ and any $U, V \in \mathcal{L}(\mathcal{B})$ we have*

$$U^n - V^n = \sum_{\gamma=1}^{n} U^{n-\gamma}(U - V)V^{\gamma-1}. \qquad (35)$$

In what follows we write $\mathbb{I}$ for the identity operator on $\mathcal{B}$ and recall that $C(t) = A(t) + B(t)$. The stability of the basic properties of the $A(t)$'s relative to the perturbation by the $B(t)$'s is stated in the following result.

**Lemma 2.** *(a) Assume that Hypotheses (A1) and (B1) hold. Then for any $t \in [0, T]$ the operator $-C(t)$ is the infinitesimal generator of a holomorphic semigroup on $\mathcal{B}$. Moreover, for every such $t$ the operator $C(t)$ is invertible and we have $C^{-1}(t) \in \mathcal{L}(\mathcal{B})$.*

*(b) Assume that Hypotheses (A1), (A2), (B1) and (B2) hold. Then the function $t \mapsto C^{-1}(t)$ is continuously differentiable with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ and there exist constants $c_2 \in \mathbb{R}^+_*$, $\tilde{c}_2 \in (0, 1]$ such that the Hölder continuity estimate*

$$\left\| \frac{dC^{-1}(t)}{dt} - \frac{dC^{-1}(s)}{ds} \right\|_\infty \leq c_2 \left| t - s \right|^{\tilde{c}_2}$$

*is valid for every $s, t \in [0, T]$.*

*(c) Assume that Hypotheses (A1), (A3), (B1) and (B3) hold. Then there exist constants $c_3 \in \mathbb{R}_*^+$, $\tilde{c}_3 \in (0,1]$ such that the inequality*

$$\left\| \frac{\partial}{\partial t} R(C(t),\lambda) \right\|_\infty \leq c_3 \, |\lambda|^{-\tilde{c}_3} \tag{36}$$

*holds for every $t \in [0,T]$ and each $\lambda \in S_\theta \diagdown \{0\}$.*

**Proof.** In order to prove (a), it is sufficient to show that $\mathcal{S}_\theta \subseteq \rho(C(t))$ and that

$$\|R(C(t),\lambda)\|_\infty \leq \tilde{c}_*(1+|\lambda|)^{-1} \tag{37}$$

for every $t \in [0,T]$ and each $\lambda \in \mathcal{S}_\theta$ for some $\tilde{c}_* \in \mathbb{R}_*^+$ independent of $t$ and $\lambda$, where $\mathcal{S}_\theta$ is given by (24). Let $\lambda \in \mathcal{S}_\theta$ and let us choose $v = R(A(t),\lambda)w$ in (26) where $w \in \mathcal{B} \diagdown \{0\}$ is arbitrary; then, by virtue of (23) and the choice of $a, b$ in Hypothesis (B1) we have

$$\|B(t)R(A(t),\lambda)w\| \leq (a + bc_*) \, \|w\| < \|w\| \, , \tag{38}$$

so that $(\mathbb{I}{+}B(t)R(A(t),\lambda))^{-1} \in \mathcal{L}(\mathcal{B})$. Therefore we get

$$\begin{aligned} R(C(t),\lambda) &= R(A(t),\lambda)(\mathbb{I}{+}B(t)R(A(t),\lambda))^{-1} \\ &= R(A(t),\lambda) \sum_{m=0}^{+\infty} (-1)^m \left(B(t)R(A(t),\lambda)\right)^m \in \mathcal{L}(\mathcal{B}), \end{aligned} \tag{39}$$

which implies (37) as a consequence of (23) and (38).

In order to prove (b), we first remark that (39) implies

$$C^{-1}(t) = A^{-1}(t)\mathsf{D}(t) \tag{40}$$

when $\lambda = 0$, where we have defined

$$\mathsf{D}(t) := (\mathbb{I}{+}B(t)A^{-1}(t))^{-1} = \sum_{m=0}^{+\infty} (-1)^m \left(B(t)A^{-1}(t)\right)^m$$

for every $t \in [0,T]$. According to Hypothesis (A2) and (40), it is then sufficient to prove that the function $t \mapsto \mathsf{D}(t)$ is continuously differentiable with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ on $[0,T]$ and that its derivative $t \mapsto \frac{d\mathsf{D}(t)}{dt}$ is Hölder continuous there; but this follows from Hypothesis (B2) and standard arguments based on the decomposition formulae (34) and (35).

The starting point for the proof of (c) is (39), which we rewrite as

$$R(C(t),\lambda) = R(A(t),\lambda)\mathsf{D}(t,\lambda)$$

by analogy with (40), where we have defined

$$\mathsf{D}(t,\lambda) := (\mathbb{I}{+}B(t)R(A(t),\lambda))^{-1} = \sum_{m=0}^{+\infty} (-1)^m \left(B(t)R(A(t),\lambda)\right)^m .$$

On the one hand, it follows from the preceding expression and (38) that the function $t \mapsto \mathsf{D}(t,\lambda)$ is bounded in the norm-topology of $\mathcal{L}(\mathcal{B})$ on $[0,T]$ uniformly in $\lambda \in S_\theta$. On the other hand, it also follows from standard arguments that the function $t \mapsto \mathsf{D}(t,\lambda)$ is continuously differentiable with respect to the norm-topology of $\mathcal{L}(\mathcal{B})$ on $[0,T]$, and that the representation

$$\begin{aligned}
&\frac{\partial \mathsf{D}(t,\lambda)}{\partial t} \\
&= \sum_{m=1}^{+\infty}(-1)^m \sum_{k=0}^{m-1} \left(B(t)R(A(t),\lambda)\right)^k \frac{\partial (B(t)R(A(t),\lambda))}{\partial t} \left(B(t)R(A(t),\lambda)\right)^{m-k-1}
\end{aligned}$$

holds as a convergent series in the Banach space of all continuous mappings from $[0,T]$ into $\mathcal{L}(\mathcal{B})$ endowed with the uniform topology. From this, (38) and Hypothesis (B3) we then infer that the estimate

$$\sup_{t\in[0,T]} \left\| \frac{\partial \mathsf{D}(t,\lambda)}{\partial t} \right\|_\infty \leq \left( \sum_{m=1}^{+\infty} m\rho^{m-1} \right) \left\| \frac{\partial (B(t)R(A(t),\lambda))}{\partial t} \right\|_\infty \leq c < +\infty$$

is valid uniformly in $\lambda \in S_\theta$. Consequently, since we have

$$\frac{\partial}{\partial t} R(C(t),\lambda) = R(A(t),\lambda) \frac{\partial \mathsf{D}(t,\lambda)}{\partial t} + \frac{\partial}{\partial t} R(A(t),\lambda) \times \mathsf{D}(t,\lambda)$$

we evidently get

$$\begin{aligned}
&\left\| \frac{\partial}{\partial t} R(C(t),\lambda) \right\|_\infty \\
&\leq c \left( \left\| R(A(t),\lambda) \right\|_\infty + \left\| \frac{\partial}{\partial t} R(A(t),\lambda) \right\|_\infty \right) \\
&\leq c \left( c_* \left(1+|\lambda|\right)^{-1} + a_3 \left|\lambda\right|^{-\tilde{a}_3} \right) \leq c_3 \left|\lambda\right|^{-\tilde{a}_3}
\end{aligned}$$

for some $c_3 \in \mathbb{R}_*^+$ for every $\lambda \in S_\theta \diagdown \{0\}$, by virtue of (23), (25) and the fact that $\tilde{a}_3 \in (0,1]$; we may thus choose $\tilde{c}_3 = \tilde{a}_3$. ■

Lemma 2 along with Tanabe's theory developed in [35] then imply the following result.

**Proposition 1.** *Assume that Hypotheses (A1)-(A3) and (B1)-(B3) hold. Then, there exists an evolution system $U_{A+B}(t,s)_{0\leq s\leq t\leq T}$ for Problem (21) such that Statement (1) of the theorem holds for every $s,t$ with $0 \leq s < t \leq T$.*

The remaining results are, therefore, preparatory statements which will lead to our proof of the product formula. From now on we may assume that $t \in (s,T)$ since Statement (2) of the theorem trivially holds for $t = s$, and begin with the following uniformity result which is the consequence of an elementary compactness argument.

**Lemma 3.** *Let* $(R(h,r))_{(h,r)\in(0,T-t]\times(s,t]} \subset \mathcal{L}(\mathcal{B})$ *be a family of operators satisfying*

$$\sup_{(h,r)\in(0,T-t]\times(s,t]} \|R(h,r)\|_\infty < +\infty. \tag{41}$$

*Furthermore let* $\mathcal{K} \subset \mathcal{B}$ *be compact, let* $I$ *be any subinterval of* $(s,t]$ *and assume that the limit*

$$\lim_{h\to 0} R(h,r)v = 0 \tag{42}$$

*exists for every* $v \in \mathcal{K}$ *in the strong topology of* $\mathcal{B}$ *uniformly in* $r \in I$*. Then, (42) holds uniformly in* $v \in \mathcal{K}$*.*

**Proof.** On the one hand, if the preceding conclusion does not hold there exist $\epsilon \in \mathbb{R}_*^+$ along with a sequence $(h_n) \subset (0, T-t]$ satisfying $h_n < \frac{1}{n}$, together with sequences $(r_n) \subset I$, $(v_n) \subset \mathcal{K}$, such that the inequality

$$\|R(h_n, r_n)v_n\| > \epsilon \tag{43}$$

holds true for every $n \in \mathbb{N}^+$. On the other hand, because of the compactness of $\mathcal{K}$ we may assume that $v_n \to v^* \in \mathcal{K}$ in the strong topology of $\mathcal{B}$ when $n \to +\infty$, so that by virtue of (41) and (42) the estimate

$$\begin{aligned}
&\|R(h_n, r_n)v_n\| \\
&\leq \|R(h_n, r_n)(v_n - v^*)\| + \|R(h_n, r_n)v^*\| \\
&\leq c\,\|v_n - v^*\| + \|R(h_n, r_n)v^*\| \leq \epsilon
\end{aligned}$$

is valid for every $n \geq N(\epsilon, v^*)$ for some $N(\epsilon, v^*) \in \mathbb{N}^+$, thereby contradicting (43). ■

We now introduce three families of linear operators on $\mathcal{B}$ whose properties will be crucial in our proof of Statement (2) below; indeed we define $E(h,r), F(h,r), G(h,r)$ by

$$\begin{aligned}
E(h,r) &= h^{-1}\left(\mathbb{I} - \exp\left[-hA(r)\right]\right) - A(r), \\
F(h,r) &= h^{-1}\exp\left[-hA(r)\right]\left(\mathbb{I} - \exp\left[-hB(r)\right]\right) - B(r), \\
G(h,r) &= h^{-1}\left(\mathbb{I} - U_{A+B}(r+h,r)\right) - C(r)
\end{aligned} \tag{44}$$

for every $(h,r) \in (0,T-t] \times (s,t]$, respectively. From these relations and the definition of the generator $C(r)$ we then obtain

$$\begin{aligned}
&E(h,r) + F(h,r) - G(h,r) \\
&= h^{-1}\left(U_{A+B}(r+h,r) - \exp\left[-hA(r)\right]\exp\left[-hB(r)\right]\right).
\end{aligned} \tag{45}$$

The following result unveils the behavior of these operators when $h \to 0$, and part of its proof is a consequence of a repeated application of Lemma 3.

**Lemma 4.** *Assume that the same hypotheses as in Proposition 1 hold; moreover, assume that Hypotheses (A4) and (D) hold, along with the $\mathcal{C}_0$-continuity and the contractive property of Hypothesis (B4). Then we have*

$$
\begin{aligned}
&\lim_{h\to 0}\sup_{r\in(s,t]}\|E(h,r)U_{A+B}(r,s)w\| \\
&=\lim_{h\to 0}\sup_{r\in(s,t]}\|F(h,r)U_{A+B}(r,s)w\| \\
&=\lim_{h\to 0}\sup_{r\in(s,t]}\|G(h,r)U_{A+B}(r,s)w\|=0
\end{aligned}
\tag{46}
$$

*for every $t\in(0,T)$ and each $w\in\mathcal{D}$.*

**Proof.** In order to prove the first two relations in (46), it is sufficient to show that the two limits

$$
\begin{aligned}
&\lim_{h\to 0}\sup_{r\in[s+\mu,t]}\|E(h,r)U_{A+B}(r,s)w\| \\
=\;&\lim_{h\to 0}\sup_{r\in[s+\mu,t]}\|F(h,r)U_{A+B}(r,s)w\|=0
\end{aligned}
\tag{47}
$$

hold uniformly in $\mu\in(0,t-s)$, respectively. From the definition of $E(h,r)$ and a general property of $\mathcal{C}_0$-semigroups we may write

$$
E(h,r)U_{A+B}(r,s)w=h^{-1}\int_0^h dk\,(\exp[-kA(r)]-\mathbb{I})\,A(r)U_{A+B}(r,s)w
$$

since $U_{A+B}(r,s)w\in\mathcal{D}(A(r))$ for every $r\in[s+\mu,t]$ and each $w\in\mathcal{D}$ as a consequence of (31). Therefore we obtain the inequality

$$
\|E(h,r)U_{A+B}(r,s)w\|\le\sup_{k\in[0,h]}\|(\exp[-kA(r)]-\mathbb{I})\,A(r)U_{A+B}(r,s)w\|,
$$

so that in order to get the first relation in (47) it is sufficient to have

$$
\lim_{h\to 0}\sup_{r\in[s+\mu,t]}\|(\exp[-hA(r)]-\mathbb{I})A(r)U_{A+B}(r,s)w\|=0 \tag{48}
$$

for every $w\in\mathcal{D}$ uniformly in $\mu$. To this end let us define $R(h,r)$ and $\mathcal{K}_\mu$ by

$$
R(h,r)=\exp[-hA(r)]-\mathbb{I} \tag{49}
$$

and

$$
\mathcal{K}_\mu=\{v\in\mathcal{B}:v=A(r)U_{A+B}(r,s)w,\ r\in[s+\mu,t]\}
$$

respectively; since the semigroup in (49) is contractive, it is clear that $R(h,r)$ satisfies (41). Furthermore we also have

$$
\lim_{h\to 0}\sup_{r\in[s+\mu,t]}R(h,r)v=0 \tag{50}
$$

for every $v \in \mathcal{B}$ uniformly in $\mu$; in fact, from (28), (49) and for every $v \in \mathcal{D}$ we obtain the estimate

$$\|R(h,r)v\| \leq \int_0^h dk \, \|\exp\left[-kA(r)\right] A(r)v\| \leq h \sup_{r\in(0,T]} \|A(r)v\| \to 0$$

as $h \to 0$ uniformly in $r$ and $\mu$, which then leads to (50) since $\mathcal{D} \subset \mathcal{B}$ is dense. Finally, in order to prove the compactness of $\mathcal{K}_\mu$ it is sufficient to prove the continuity of the mapping $r \mapsto A(r)U_{A+B}(r,s)w$ in the strong topology of $\mathcal{B}$ for $r \in [s+\mu, t]$; but this is an immediate consequence of Hypothesis (B2), (40) and Statement (1) of the theorem since we have

$$\begin{aligned} & A(r)U_{A+B}(r,s)w \\ = \; & \left(\mathbb{I} - B(r)A^{-1}(r)\mathsf{D}(r)\right)(A(r)+B(r))U_{A+B}(r,s)w. \end{aligned}$$

Therefore, (48) indeed emerges as a consequence of Lemma 3.

We now proceed in much the same way to prove the second relation in (47). We start with the integral representation

$$\begin{aligned} & F(h,r)U_{A+B}(r,s)w \\ = \; & h^{-1}\int_0^h dk \left(\exp\left[-hA(r)\right]\exp\left[-kB(r)\right] - \mathbb{I}\right) B(r)U_{A+B}(r,s)w \end{aligned}$$

valid for every $w \in \mathcal{D}$, which leads to the inequality

$$\begin{aligned} & \|F(h,r)U_{A+B}(r,s)w\| \\ \leq \; & \sup_{k\in[0,h]} \|\left(\exp\left[-hA(r)\right]\exp\left[-kB(r)\right] - \mathbb{I}\right) B(r)U_{A+B}(r,s)w\|. \end{aligned} \tag{51}$$

This time we define $R(h,r)$ and $\mathcal{K}_\mu$ by

$$R(h,r) = \exp\left[-hA(r)\right]\exp\left[-kB(r)\right] - \mathbb{I} \tag{52}$$

and

$$\mathcal{K}_\mu = \{v \in \mathcal{B} : v = B(r)U_{A+B}(r,s)w, \; r \in [s+\mu, t]\}$$

respectively; from the contractive properties of the semigroups involved it follows again that (52) satisfies (41), while the relation

$$\lim_{h\to 0} \sup_{r\in[s+\mu,t]} R(h,r)v = 0 \tag{53}$$

is in this case a consequence of the identity

$$R(h,r) = \exp\left[-hA(r)\right]\left(\exp\left[-kB(r)\right] - \mathbb{I}\right) + \exp\left[-hA(r)\right] - \mathbb{I}.$$

In fact, by using the same kind of integral representation as above along with (29) we obtain

$$\begin{aligned} & \|\exp\left[-hA(r)\right]\left(\exp\left[-kB(r)\right] - \mathbb{I}\right)v\| \\ \leq \; & \int_0^k dl \, \|\exp\left[-lB(r)\right] B(r)v\| \leq h \sup_{r\in(0,T]} \|B(r)v\| \to 0 \end{aligned} \tag{54}$$

as $h \to 0$ uniformly in $r$ and $\mu$ for every $v \in \mathcal{D}$ and hence for every $v \in \mathcal{B}$, while from the first part of the proof we get

$$\sup_{r\in[s+\mu,t]} \|(\exp[-hA(r)] - \mathbb{I})\, v\| \to 0 \tag{55}$$

as $h \to 0$ for every $v \in \mathcal{B}$, both (54) and (55) implying (53) for every such $v$. Finally, for the same reasons as above $\mathcal{K}_\mu$ is compact, so that from Lemma 3 we infer that the relation

$$\lim_{h\to 0} \sup_{r\in[s+\mu,t]} \|(\exp[-hA(r)] \exp[-kB(r)] - \mathbb{I})\, B(r)U_{A+B}(r,s)w\| = 0$$

holds for every $w \in \mathcal{D}$ uniformly in $\mu$; together with (51) this implies the desired result.

The proof of the third relation in (46) follows a slightly different line, the starting point being the identity

$$\begin{aligned} & G(h,r)U_{A+B}(r,s)w \\ = \quad & h^{-1}\int_r^{r+h} dk\,(C(k)U_{A+B}(k,s)w - C(r)U_{A+B}(r,s)w) \end{aligned} \tag{56}$$

which is a simple consequence of Statement (1) of the theorem and of the third relation in (44). Since that statement also implies the *uniform* continuity of the function $k \to C(k)U_{A+B}(k,s)w - C(r)U_{A+B}(r,s)w$ with respect to the strong topology of $\mathcal{B}$ on the compact interval $[r, r+h]$, we may conclude that for every $\epsilon \in \mathbb{R}_*^+$ there exists an $h_\epsilon \in \mathbb{R}_*^+$ such that the inequalities $0 \le k - r \le h \le h_\epsilon$ together with (56) lead to the estimate

$$\|G(h,r)U_{A+B}(r,s)w\| \le \epsilon$$

uniformly in $r \in (s,t]$, which is the desired result. ■

Finally, we will still need the following continuity result.

**Lemma 5.** *Assume that Hypotheses (A1), (A2), (A4), (B4) and (D) hold. Then, the functions*

$$r \mapsto \exp[-(r-s)A(r)] \tag{57}$$

*and*

$$r \mapsto \exp[-(r-s)B(r)] \tag{58}$$

*are continuous on the interval $[s,t]$ in the strong topology of $\mathcal{L}(\mathcal{B})$.*

**Proof.** The proof of (57) follows from the standard arguments of [35]. As for (58) we first prove the right-continuity at $r = s$; for this we start by noticing that the analyticity part of Hypothesis (B4) allows us to write

$$\exp[-(r-s)B(r)]\, v = \frac{1}{2\pi i}\int_{\Gamma_{r,s}} d\lambda e^{-(r-s)\lambda} R(B(r),\lambda)v \tag{59}$$

for every $v \in \mathcal{B}$ and each $r \in (s,t]$, where $\Gamma_{r,s} := \Gamma_{1,r,s} \cup \Gamma_{2,r,s} \cup \Gamma_{3,r,s}$ is the union of the three paths

$$\Gamma_{1,r,s} = \left\{ \mu e^{-i\theta^*} : \frac{1}{r-s} \leq \mu < +\infty \right\},$$

$$\Gamma_{2,r,s} = \left\{ \frac{1}{r-s} e^{i\mu} : \theta^* \leq \mu \leq 2\pi - \theta^* \right\}$$

and

$$\Gamma_{3,r,s} = \left\{ \mu e^{i\theta^*} : \frac{1}{r-s} \leq \mu < +\infty \right\}.$$

The orientation we choose for $\Gamma_{r,s}$ is that of increasing values of $\operatorname{Im}\lambda$. From the residue theorem and the chosen orientation of $\Gamma_{r,s}$ we then easily obtain

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{r,s}} d\lambda e^{-(r-s)\lambda} \lambda^{-1} = -1,$$

so that we may write

$$\begin{aligned}
& \exp\left[-(r-s)B(r)\right] v - v \\
= \; & \frac{1}{2\pi i} \int\limits_{\Gamma_{r,s}} d\lambda e^{-(r-s)\lambda} \left( R(B(r),\lambda) + \lambda^{-1} \right) v \\
= \; & \frac{1}{2\pi i} \int\limits_{\Gamma_{r,s}} d\lambda e^{-(r-s)\lambda} \lambda^{-1} R(B(r),\lambda) B(r) v \\
= \; & \frac{1}{2\pi i} \int\limits_{\Gamma'} d\lambda e^{-\lambda} \lambda^{-1} R\left( B(r), \frac{\lambda}{r-s} \right) B(r) v
\end{aligned} \tag{60}$$

for every $v \in \mathcal{D}$, the dense set of Hypothesis (D), where the new integration path $\Gamma' := (r-s)\Gamma_{r,s}$ in (60) is *independent of* $r$ and $s$. Therefore, owing to (29) and (30) we obtain

$$\begin{aligned}
& \left\| \exp\left[-(r-s)B(r)\right] v - v \right\| \\
\leq \; & c \sup_{r \in (0,T]} \left\| B(r) v \right\| (r-s) \int\limits_{\Gamma'} |d\lambda| \left| e^{-\lambda} \right| |\lambda|^{-2} \to 0
\end{aligned}$$

as $r \searrow s$; since $\mathcal{D}$ is dense and the semigroup $\exp\left[-sB(t)\right]_{s \geq 0}$ contractive, the desired right-continuity at $r = s$ follows.

As for the proof of the continuity away from the left endpoint, we find it more convenient to choose the integration path $\Gamma := \Gamma_1 \cup \Gamma_2$ in (59) rather than the path $\Gamma_{r,s}$, where

$$\Gamma_1 = \left\{ \mu e^{-i\theta^*} : 0 \leq \mu < +\infty \right\}$$

and

$$\Gamma_2 = \left\{ \mu e^{i\theta^*} : 0 \leq \mu < +\infty \right\},$$

the orientation being the same as before. Let us fix $r \in (s,t)$, let $(r_n)_{n\in\mathbb{N}^+}$ be any sequence such that $r_n > r$ with $r_n \to r$ as $n \to +\infty$ and write

$$\begin{aligned}
& \exp\left[-(r-s)B(r)\right] v - \exp\left[-(r_n-s)B(r_n)\right] v \\
= & \frac{1}{2\pi i}\int_\Gamma d\lambda e^{-(r-s)\lambda}\left(R(B(r),\lambda) - R(B(r_n),\lambda)\right) v \\
& + \frac{1}{2\pi i}\int_\Gamma d\lambda e^{-(r-s)\lambda}\left(1 - e^{-(r_n-r)\lambda}\right) R(B(r_n),\lambda) v. \qquad (61)
\end{aligned}$$

From the continuity part of Hypothesis (B4) we have

$$\lim_{n\to+\infty} \sup_{\lambda\in S_{\theta^*}} \left\|\left(R(B(r),\lambda) - R(B(r_n),\lambda)\right) v\right\| = 0,$$

which indeed implies that

$$\lim_{n\to+\infty}\int_\Gamma d\lambda e^{-(r-s)\lambda}\left(R(B(r),\lambda) - R(B(r_n),\lambda)\right) v = 0 \qquad (62)$$

strongly in $\mathcal{B}$ for every $v$ since

$$\int_\Gamma |d\lambda| \left|e^{-(r-s)\lambda}\right| < +\infty.$$

Furthermore, the norm of the integrand in the second term on the right-hand side of (61) goes to zero as $n \to +\infty$ for every $\lambda \in \Gamma$. Moreover, owing to (30) and to our choice of the $r_n$'s we can estimate that norm as

$$\begin{aligned}
& \left\| e^{-(r-s)\lambda}\left(1 - e^{-(r_n-r)\lambda}\right) R(B(r_n),\lambda) v \right\| \\
\leq & \; c\frac{\left|e^{-(r-s)\lambda}\right|}{1+|\lambda|}\|v\|
\end{aligned}$$

uniformly in $n$, so that we eventually get

$$\lim_{n\to+\infty}\int_\Gamma d\lambda e^{-(r-s)\lambda}\left(1 - e^{-(r_n-r)\lambda}\right) R(B(r_n),\lambda) v = 0 \qquad (63)$$

strongly in $\mathcal{B}$ for every $v$ by dominated convergence since

$$\int_\Gamma |d\lambda| \frac{\left|e^{-(r-s)\lambda}\right|}{1+|\lambda|} < +\infty.$$

A similar argument holds for $r \in (s,t]$ if $(r_n)_{n\in\mathbb{N}^+}$ is any sequence such that $r_n < r$ with $r_n \to r$ as $n \to +\infty$. This, together with (61)-(63) proves the desired continuity away from $r = s$.

Evidently, if the $B$'s are independent of $r$ the $\mathcal{C}_0$-continuity of $\exp\left[-sB\right]_{s\geq 0}$ alone gives the result. ■

We are now ready for the following.

**Proof of the theorem.** By virtue of Proposition 1 it remains to prove Statement (2). For every $n \in \mathbb{N}^+$ sufficiently large we set $h = \frac{t-s}{n}$ and define the sequence of products $(P_n(t,s)) \subset \mathcal{L}(\mathcal{B})$ by

$$P_n(t,s) = U_{A+B}(t,s) - \prod_{\gamma=n}^{1} \exp\left[-hA\left(s+(\gamma-1)h\right)\right]\exp\left[-hB\left(s+(\gamma-1)h\right)\right].$$

Since both $\exp\left[-sA(t)\right]_{s\geq 0}$ and $\exp\left[-sB(t)\right]_{s\geq 0}$ are semigroups of contractions for every $t \in [0,T]$ the sequence $(P_n(t,s))$ is bounded in $\mathcal{L}(\mathcal{B})$, so that in order to prove the product formula it is sufficient to show that $P_n(t,s)v \to 0$ as $n \to +\infty$ in the strong topology of $\mathcal{B}$ for every $v \in \mathcal{D}$, the dense set of Hypothesis (D). To this end we choose the two families $(U_\gamma)_{\gamma\in\{1,\ldots,n\}}$, $(V_\gamma)_{\gamma\in\{1,\ldots,n\}}$ of Lemma 1 as

$$\begin{aligned}
U_\gamma &= U_{A+B}(s+\gamma h, s+(\gamma-1)h),\\
V_\gamma &= \exp\left[-hA\left(s+(\gamma-1)h\right)\right]\exp\left[-hB\left(s+(\gamma-1)h\right)\right],
\end{aligned}$$

respectively; owing to the basic composition law of the $U_{A+B}(t,s)$'s and by virtue of Lemma 1 we then have after some rearrangements

$$\begin{aligned}
P_n(t,s) \quad &= \quad -\sum_{\gamma=1}^{n-1}\prod_{\alpha=n}^{\gamma+1} V_\alpha \times \left(V_\gamma - U_\gamma\right) U_{A+B}(s+(\gamma-1)h, s)\\
&\quad\quad +(U_n - V_n)U_{A+B}(s+(n-1)h, s).
\end{aligned}$$

Therefore, for every $v \in \mathcal{D}$ and by using again the estimate

$$\left\|\prod_{\alpha=n}^{\gamma+1} V_\alpha\right\|_\infty \leq 1$$

we obtain the inequalities

$$\begin{aligned}
&\|P_n(t,s)v\| \leq \sum_{\gamma=1}^{n} \|(U_\gamma - V_\gamma)U_{A+B}(s+(\gamma-1)h,s)v\|\\
&\leq n \sup_{r\in[s,s+(n-1)h]} \|U_{A+B}(r+h,s)v - \exp\left[-hA(r)\right]\exp\left[-hB(r)\right]U_{A+B}(r,s)v\|\\
&\leq n \sup_{r\in[s,t]} \|U_{A+B}(r+h,s)v - \exp\left[-hA(r)\right]\exp\left[-hB(r)\right]U_{A+B}(r,s)v\| \qquad (64)
\end{aligned}$$

after a simple change of summation variable, where we may now write $r = s + \kappa nh$ for $\kappa \in [0,1]$. Furthermore, expressing $h$ in (64) as a function of $r$ by means of this last relation and by using Lemma 5, we see that the function

$$r \mapsto U_{A+B}\left(r + \frac{r-s}{\kappa n}, s\right) - \exp\left[-\frac{r-s}{\kappa n}A(r)\right]\exp\left[-\frac{r-s}{\kappa n}B(r)\right]U_{A+B}(r,s)$$

is continuous on the interval $[s,t]$ in the strong topology of $\mathcal{L}(\mathcal{B})$ since the evolution system $U_{A+B}(t,s)_{0\leq s\leq t\leq T}$ also enjoys this property. Consequently, this allows us to replace the interval $[s,t]$ by $(s,t]$ in (64), so that we finally obtain

$$\begin{aligned} & \|P_n(t,s)v\| \\ \leq \; & c \sup_{r\in(s,t]}\left\|\left(E\left(\frac{r-s}{\kappa n}, r\right) + F\left(\frac{r-s}{\kappa n}, r\right) - G\left(\frac{r-s}{\kappa n}, r\right)\right)U_{A+B}(r,s)v\right\| \to 0 \end{aligned}$$

for every $v \in \mathcal{D}$ when $n \to +\infty$, as a consequence of (45) and Lemma 4. ■

We devote the next section to the discussion of some examples illustrating the statements of our main theorem.

# 4 Some Simple Examples

While it is clear that our theorem has a wide range of potential applications, we shall restrict ourselves here to the simplest situations. We first consider a particular case of (10), namely, the class of parabolic initial-value problems given by

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \operatorname{div}(k(x,t)\nabla u(x,t)) - (\varkappa + \varepsilon m(x,t))\,u(x,t), \quad (x,t) \in D \times (0,T], \\ u(x,0) &= u_0(x), \quad x \in D, \\ \frac{\partial u(x,t)}{\partial n(k)} &= 0, \quad (x,t) \in \partial D \times (0,T]. \end{aligned} \tag{65}$$

In this case (13) reduces to

$$\mathsf{b}_\varepsilon(t,v,w) = \varepsilon \int_D dx m(x,t) v(x) \overline{w}(x)$$

and thus extends to a Hermitian sesquilinear form on $\mathcal{B} = L^2(D,\mathbb{C})$, so that the associated multiplication operator $B_\varepsilon(t) := \varepsilon B(t)$ is bounded and self-adjoint there. In order for (65) to fit the theory of the preceding section, however, we need to impose stronger conditions on the coefficients than (K) and (M) do. The following smoothness requirements are sufficient for this purpose.

(K′) We have $k : \overline{D} \times [0,T] \mapsto \mathbb{R}^{d^2}$ and for every $i,j \in \{1,...,d\}$ the functions $(x,t) \mapsto k_{i,j}(x,t) = k_{j,i}(x,t)$ are continuously differentiable on $\overline{D}\times [0,T]$;

moreover, the ellipticity condition (3) holds and there exist constants $c'_* \in \mathbb{R}^+_*$, $\sigma' \in (0,1]$ such that the Hölder continuity estimate

$$\max_{i,j\in\{1,\ldots,d\}} \left| \frac{\partial k_{i,j}(x,t)}{\partial t} - \frac{\partial k_{i,j}(x,s)}{\partial s} \right| \leq c'_* \left|t-s\right|^{\sigma'} \tag{66}$$

is valid for every $x \in \overline{D}$ and every $s,t \in [0,T]$.

(M′) We have $m \in L^\infty(\overline{D} \times [0,T], \mathbb{R}^+)$ and $t \mapsto m(x,t)$ is continuously differentiable on $[0,T]$ uniformly in $x \in \overline{D}$ with $\frac{\partial m}{\partial t} \in L^\infty(\overline{D} \times [0,T], \mathbb{R})$; moreover, the Hölder continuity estimate

$$\left| \frac{\partial m(x,t)}{\partial t} - \frac{\partial m(x,s)}{\partial s} \right| \leq c'_* \left|t-s\right|^{\sigma'}$$

holds for every $x \in \overline{D}$ and every $s,t \in [0,T]$.

We then have the following result.

**Proposition 2.** *Assume that Hypotheses (K′) and (M′) hold; then, all the conclusions of the theorem are valid for the evolution system given by (17). In particular, for all $s,t$ with $0 \leq s \leq t < T$ and every $\varepsilon \in \mathbb{R}^+$ sufficiently small we have*

$$\begin{aligned} &U_{A+B_\varepsilon}(t,s) \\ = \quad &\lim_{n\to+\infty} \prod_{\gamma=n-1}^{0} \exp\left[-\frac{t-s}{n} A\left(s + \frac{\gamma}{n}(t-s)\right)\right] \exp\left[-\frac{t-s}{n} B_\varepsilon\left(s + \frac{\gamma}{n}(t-s)\right)\right] \end{aligned} \tag{67}$$

*in the strong topology of $\mathcal{L}(L^2(D,\mathbb{C}))$, where $\exp\left[-sA(t)\right]_{s\geq 0}$ and $\exp\left[-sB_\varepsilon(t)\right]_{s\geq 0}$ are the semigroups generated by (7) and $-B_\varepsilon(t)$, respectively. Thus, the reconstruction formula (33) also holds in this case.*

The proof of Proposition 2 rests on several lemmas and remarks. Without restricting the generality, we first choose $\varkappa \geq \underline{k}$ in (65). Then, the operator given by (7) satisfies Hypotheses (A1), (A4) as an immediate consequence of standard Lax-Milgram arguments and elliptic regularity theory; moreover, Hypothesis (B1) trivially holds for $\varepsilon$ sufficiently small since $B_\varepsilon(t)$ is bounded. The verification of the remaining hypotheses requires more work; we settle the question regarding (A2) with the following result.

**Lemma 6.** *Assume that Hypothesis (K′) holds; then, the function $t \mapsto A^{-1}(t)f$ is strongly differentiable in $L^2(D,\mathbb{C})$ for every $f$; moreover, we have $\frac{d}{dt}A^{-1}(t)f \in H^1(D,\mathbb{C})$ and there exists a constant $a_2 \in \mathbb{R}^+_*$ such that the Hölder continuity estimate*

$$\left\| \frac{d}{dt}A^{-1}(t)f - \frac{d}{ds}A^{-1}(s)f \right\|_{1,2} \leq a_2 \left|t-s\right|^{\sigma'} \left\|f\right\|_2 \tag{68}$$

*holds for all $s, t \in [0, T]$ and every $f \in L^2(D, \mathbb{C})$, with $\sigma'$ the Hölder exponent in (66).*

**Proof.** Let us write $u(s) := A^{-1}(s)f$ with $u(s) \in \mathcal{D}(A(s))$; from (8) we then have

$$\mathsf{a}(s, u(s), w) + \varkappa(u(s), w)_2 = (f, w)_2 \tag{69}$$

and

$$\mathsf{a}(t, u(t), w) + \varkappa(u(t), w)_2 = (f, w)_2 \tag{70}$$

for all $s, t \in [0, T]$ and every $w \in H^1(D, \mathbb{C})$, so that by subtracting (69) from (70) we obtain

$$\begin{aligned} & \mathsf{a}(t, u(t) - u(s), w) \\ & + \varkappa\,(u(t) - u(s), w)_2 \\ = \; & \mathsf{a}(s, u(s), w) - \mathsf{a}(t, u(s), w). \end{aligned} \tag{71}$$

Next, for every $t \in [0, T]$ we introduce the shorthand notation

$$(v, w)_{1,2,t} := \mathsf{a}(t, v, w) + \varkappa(v, w)_2 \tag{72}$$

and observe that these new sesquilinear forms defined on $H^1(D, \mathbb{C}) \times H^1(D, \mathbb{C})$ induce norms $\|.\|_{1,2,t}$ equivalent to $\|.\|_{1,2}$ on $H^1(D, \mathbb{C})$ by virtue of the boundedness of the $k_{i,j}$'s and (3), the equivalence constants being independent of $t$. Then, fix $t$ and set $h = s - t$ in (71); owing to (4) and (72) this gives

$$\begin{aligned} & \left( \frac{u(t+h) - u(t)}{h}, w \right)_{1,2,t} \\ = \; & -\sum_{i,j=1}^{d} \int_D dx \frac{k_{i,j}(x, t+h) - k_{i,j}(x, t)}{h} u_{x_i}(x, t+h) \overline{w}_{x_j}(x), \end{aligned} \tag{73}$$

from which equality we now want to prove that

$$\begin{aligned} & \lim_{h \to 0} \left( \frac{u(t+h) - u(t)}{h}, w \right)_{1,2,t} \\ = \; & -\sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x, t)}{\partial t} u_{x_i}(x, t) \overline{w}_{x_j}(x). \end{aligned} \tag{74}$$

Subtracting the right-hand side of (74) from (73) we have

$$\begin{aligned} & \left| \left( \frac{u(t+h) - u(t)}{h}, w \right)_{1,2,t} + \sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x, t)}{\partial t} u_{x_i}(x, t) \overline{w}_{x_j}(x) \right| \\ \leq \; & \sum_{i,j=1}^{d} \int_D dx \left| \frac{k_{i,j}(x, t+h) - k_{i,j}(x, t)}{h} - \frac{\partial k_{i,j}(x, t)}{\partial t} \right| |u_{x_i}(x, t)| \left| w_{x_j}(x) \right| \\ & + \sum_{i,j=1}^{d} \int_D dx \left| \frac{k_{i,j}(x, t+h) - k_{i,j}(x, t)}{h} \right| |u_{x_i}(x, t+h) - u_{x_i}(x, t)| \left| w_{x_j}(x) \right| \end{aligned}$$

and proceed by showing that these two terms each go to zero when $h \to 0$. For the first one this follows from Hypothesis (K$'$) and dominated convergence since the estimate

$$\begin{aligned} & \left| \frac{k_{i,j}(x,t+h) - k_{i,j}(x,t)}{h} - \frac{\partial k_{i,j}(x,t)}{\partial t} \right| |u_{x_i}(x,t)| \left| w_{x_j}(x) \right| \\ \leq \quad & c \, |u_{x_i}(x,t)| \left| w_{x_j}(x) \right| \end{aligned}$$

holds uniformly in $h$ as a consequence of the boundedness of $\frac{\partial k_{i,j}(x,t)}{\partial t}$ and the uniform Lipschitz continuity of $t \mapsto k_{i,j}(x,t)$. For the second one we have

$$\begin{aligned} & \sum_{i,j=1}^{d} \int_D dx \left| \frac{k_{i,j}(x,t+h) - k_{i,j}(x,t)}{h} \right| |u_{x_i}(x,t+h) - u_{x_i}(x,t)| \left| w_{x_j}(x) \right| \\ \leq \quad & c \, \|u(t+h) - u(t)\|_{1,2,t} \, \|w\|_{1,2,t} \, , \end{aligned}$$

so that it remains to prove the relation

$$\lim_{h \to 0} \|u(t+h) - u(t)\|_{1,2,t} = 0. \tag{75}$$

But going back to (73) we have

$$\begin{aligned} & \left| (u(t+h) - u(t), w)_{1,2,t} \right| \\ \leq \quad & c \, |h| \left\| A^{-1}(t+h) f \right\|_{1,2,t} \|w\|_{1,2,t} \leq c \, |h| \, \|f\|_2 \, \|w\|_{1,2,t} \end{aligned}$$

from which (75) follows immediately, thereby completing the proof of (74). Therefore, there exists $\frac{du(t)}{dt} \in H^1(D, \mathbb{C})$ such that

$$\left( \frac{du(t)}{dt}, w \right)_{1,2,t} = - \sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,t)}{\partial t} u_{x_i}(x,t) \overline{w}_{x_j}(x)$$

for every $t \in [0, T]$ and every $w \in H^1(D, \mathbb{C})$, which implies that

$$\lim_{h \to 0} \frac{u(t+h) - u(t)}{h} = \frac{du(t)}{dt}$$

strongly in $L^2(D, \mathbb{C})$ by virtue of the compact embedding $H^1(D, \mathbb{C}) \hookrightarrow L^2(D, \mathbb{C})$. Hence, the function $t \mapsto A^{-1}(t) f$ is indeed strongly differentiable in $L^2(D, \mathbb{C})$ with $\frac{d}{dt} A^{-1}(t) f \in H^1(D, \mathbb{C})$ for every $f$; furthermore, we have

$$\begin{aligned} & \left( \frac{du(t)}{dt} - \frac{du(s)}{ds}, w \right)_{1,2,t} \\ = \quad & - \sum_{i,j=1}^{d} \int_D dx \left( \frac{\partial k_{i,j}(x,t)}{\partial t} - \frac{\partial k_{i,j}(x,s)}{\partial s} \right) u_{x_i}(x,t) \overline{w}_{x_j}(x) \\ & - \sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,s)}{\partial s} \left( u_{x_i}(x,t) - u_{x_i}(x,s) \right) \overline{w}_{x_j}(x), \end{aligned}$$

which, together with (66) and arguments similar to those we just used, leads to the estimate

$$\begin{aligned} & \left| \left( \frac{du(t)}{dt} - \frac{du(s)}{ds}, w \right)_{1,2,t} \right| \\ \leq \quad & c\, |t-s|^{\sigma'} \, \|f\|_2 \, \|w\|_{1,2,t} \end{aligned}$$

and thereby to (68). ■

It is plain that the preceding construction implies the existence of a linear bounded operator $\frac{dA^{-1}(t)}{dt} : L^2(D,\mathbb{C}) \mapsto H^1(D,\mathbb{C})$ satisfying $\frac{du(t)}{dt} = \frac{dA^{-1}(t)}{dt} f$, so that the validity of Hypothesis (A2) indeed emerges as a direct consequence of Lemma 6.

We now turn to Hypothesis (A3), whose verification rests on the following result.

**Lemma 7.** *Assume that Hypothesis (K′) holds; then, the function* $t \mapsto R(A(t),\lambda)f$ *is strongly differentiable in* $L^2(D,\mathbb{C})$ *for every* $f$*; moreover, we have* $\frac{\partial}{\partial t} R(A(t),\lambda)f \in H^1(D,\mathbb{C})$ *and there exists a constant* $a_3 \in \mathbb{R}^+_*$ *such that the estimate*

$$\left\| \frac{\partial}{\partial t} R(A(t),\lambda) f \right\|_{1,2} \leq a_3 \, |\lambda|^{-\frac{1}{2}} \, \|f\|_2 \tag{76}$$

*holds for every* $t \in [0,T]$*, every* $f \in L^2(D,\mathbb{C})$ *and every* $\lambda \in \mathcal{S}_\theta \diagdown \{0\}$.

**Proof.** Let us fix $\lambda \in \rho(A(t))$; it is easy to prove the strong differentiability of $t \mapsto R(A(t),\lambda)f$ by relating $R(A(t),\lambda)f$ to $A^{-1}(t)f$ by means of the resolvent identity; we obtain

$$\frac{\partial}{\partial t} R(A(t),\lambda) f = \left(\mathbb{I} + \lambda R(A(t),\lambda)\right) \frac{dA^{-1}(t)}{dt} \left(\mathbb{I} + \lambda R(A(t),\lambda)\right) f \tag{77}$$

for every $t \in [0,T]$ and every $f \in L^2(D,\mathbb{C})$. Furthermore, (77) and the definition of $\frac{dA^{-1}(t)}{dt}$ give $\frac{\partial}{\partial t} R(A(t),\lambda)f \in H^1(D,\mathbb{C})$. Let us now fix $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\lambda \in \mathcal{S}_\theta \diagdown \{0\}$ in order to prove (76). For this we rely again on the variational structure of the problem; writing $u(t,\lambda) := R(A(t),\lambda)f$ with $u(t,\lambda) \in \mathcal{D}(A(t))$ and arguing exactly as in the proof of Lemma 6 we eventually get the relation

$$\begin{aligned} & a\left(t, \frac{\partial u(t,\lambda)}{\partial t}, w\right) + \varkappa \left(\frac{\partial u(t,\lambda)}{\partial t}, w\right)_2 - \lambda \left(\frac{\partial u(t,\lambda)}{\partial t}, w\right)_2 \\ = \quad & -\sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,t)}{\partial t} u_{x_i}(x,t,\lambda) \overline{w}_{x_j}(x) \end{aligned}$$

valid for every $t \in [0,T]$ and every $w \in H^1(D,\mathbb{C})$, which reduces to

$$
\begin{aligned}
& a\left(t, \frac{\partial u(t,\lambda)}{\partial t}, \frac{\partial u(t,\lambda)}{\partial t}\right) + \varkappa \left\| \frac{\partial u(t,\lambda)}{\partial t} \right\|_2^2 - \lambda \left\| \frac{\partial u(t,\lambda)}{\partial t} \right\|_2^2 \\
= & -\sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,t)}{\partial t} u_{x_i}(x,t,\lambda) \left(\frac{\partial \bar{u}(x,t,\lambda)}{\partial t}\right)_{x_j} \qquad (78)
\end{aligned}
$$

by choosing $w = \frac{\partial u(t,\lambda)}{\partial t}$. We first prove (76) for $\arg\lambda \geq \theta$ with $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda > 0$; for this we take the real and imaginary parts of (78) to obtain

$$
\begin{aligned}
& \operatorname{Re}\lambda \left\| \frac{\partial u(t,\lambda)}{\partial t} \right\|_2^2 \\
= & \; a\left(t, \frac{\partial u(t,\lambda)}{\partial t}, \frac{\partial u(t,\lambda)}{\partial t}\right) + \varkappa \left\| \frac{\partial u(t,\lambda)}{\partial t} \right\|_2^2 \\
& + \operatorname{Re} \sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,t)}{\partial t} u_{x_i}(x,t,\lambda) \left(\frac{\partial \bar{u}(x,t,\lambda)}{\partial t}\right)_{x_j} \qquad (79)
\end{aligned}
$$

and

$$
\operatorname{Im}\lambda \left\| \frac{\partial u(t,\lambda)}{\partial t} \right\|_2^2 = \operatorname{Im} \sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,t)}{\partial t} u_{x_i}(x,t,\lambda) \left(\frac{\partial \bar{u}(x,t,\lambda)}{\partial t}\right)_{x_j}, \qquad (80)
$$

respectively. From (79), (80), the fact that the form $a+\varkappa$ is coercive on $H^1(D,\mathbb{C})$ and from the boundedness of the $\frac{\partial k_{i,j}}{\partial t}$'s we then get

$$
\begin{aligned}
\underline{k} \left\| \frac{\partial u(t,\lambda)}{\partial t} \right\|_{1,2}^2 \leq & \; \frac{1}{\tan\theta} \operatorname{Im} \sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,t)}{\partial t} u_{x_i}(x,t,\lambda) \left(\frac{\partial \bar{u}(x,t,\lambda)}{\partial t}\right)_{x_j} \\
& - \operatorname{Re} \sum_{i,j=1}^{d} \int_D dx \frac{\partial k_{i,j}(x,t)}{\partial t} u_{x_i}(x,t,\lambda) \left(\frac{\partial \bar{u}(x,t,\lambda)}{\partial t}\right)_{x_j} \\
\leq & \; c \left\| u(t,\lambda) \right\|_{1,2} \left\| \frac{\partial u(t,\lambda)}{\partial t} \right\|_{1,2},
\end{aligned}
$$

that is,

$$
\left\| \frac{\partial}{\partial t} R(A(t),\lambda) f \right\|_{1,2} \leq c \left\| R(A(t),\lambda) f \right\|_{1,2} \qquad (81)
$$

for every $t \in [0,T]$ and every $f \in L^2(D,\mathbb{C})$. But from standard estimates for the resolvent of time-dependent sectorial operators (see, for instance, [35]) we have in this case

$$
\left\| R(A(t),\lambda) f \right\|_{1,2} \leq c \left| \lambda \right|^{-\frac{1}{2}} \left\| f \right\|_2 \qquad (82)
$$

so that (76) indeed follows from (81) and (82). The proof of (76) when $\arg\lambda \leq -\theta$ with $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda < 0$, or when $\operatorname{Re}\lambda \leq 0$ with $\lambda \neq 0$, follows from similar arguments and is thereby omitted. ■

It remains to verify Hypotheses (B2)-(B4) and (D). As far as (B2) and (B3) are concerned, it is sufficient to prove that the function $t \mapsto B(t)$ is continuously differentiable with respect to the norm-topology of $\mathcal{L}(L^2(D,\mathbb{C}))$ and that its derivative $\frac{dB(t)}{dt}$ is Hölder continuous there, for then the result follows from Lemma 6 and (77), respectively; but these required properties of $B(t)$ are immediate consequences of Hypothesis (M$'$).

As for Hypothesis (B4), the semigroup generated by $-B(t)$ is the multiplication operator

$$\exp\left[-sB(t)\right]f = \exp\left[-sm(.,t)\right]f$$

on $L^2(D,\mathbb{C})$ and is clearly holomorphic and contractive since $B(t)$ is self-adjoint and $m(.,t) \geq 0$ for every $t \in [0,T]$; consequently, the only point that requires attention is the continuity of $t \mapsto R(B(t),\lambda)$, although we can easily establish the continuity of (58) directly in this case since the $B(t)$'s are bounded. However, we wish to present an independent argument which easily carries over to the case of certain unbounded $B(t)$'s. For this we assume without restricting the generality that $\underline{m} := \inf_{(x,t)\in\overline{D}\times[0,T]} m(x,t) > 0$.

**Lemma 8.** *The mapping $t \mapsto R(B(t),\lambda)$ is Lipschitz continuous on $[0,T]$ in the norm-topology of $\mathcal{L}(\mathcal{B})$ uniformly in $\lambda \in S_{\theta^*}$ for every $\theta^* \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$.*

**Proof.** Let us write $\operatorname{Ran} m$ for the range of $m$; if $\lambda \in \mathbb{C}\diagdown\overline{\operatorname{Ran} m}$ then from the relation

$$R(B(t),\lambda)f(x) = \frac{f(x)}{m(x,t)-\lambda}$$

and the fact that $t \mapsto m(x,t)$ is Lipschitz continuous uniformly in $x$ as a consequence of Hypothesis (M$'$) we readily obtain

$$\left\|R(B(t),\lambda)f - R(B(s),\lambda)f\right\|_2 \leq c\frac{|t-s|}{d_{m,\lambda}^2}\left\|f\right\|_2 \tag{83}$$

for every $f \in L^2(D,\mathbb{C})$ and every $s,t \in [0,T]$, where

$$d_{m,\lambda} := \inf_{(x,t)\in\overline{D}\times[0,T]} |m(x,t)-\lambda| > 0$$

is the distance between $\lambda$ and $\operatorname{Ran} m$. In order to get the desired uniformity in (83), it is thus sufficient to prove that

$$d_m := \inf_{\lambda\in S_{\theta^*}} d_{m,\lambda} > 0. \tag{84}$$

Let us fix $\theta^* \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$; we first prove (84) for $\arg\lambda \geq \theta^*$ with $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda > 0$. On the one hand, if $\operatorname{Re}\lambda \in (0,\underline{m})$ we have

$$\begin{aligned} d_{m,\lambda}^2 &\geq (\operatorname{Im}\lambda)^2 + (\underline{m} - \operatorname{Re}\lambda)^2 \\ &\geq (\operatorname{Re}\lambda)^2\tan^2\theta^* + \underline{m}^2 - 2\underline{m}\operatorname{Re}\lambda \\ &\geq (\operatorname{Re}\lambda)^2\left(\tan^2\theta^* - \epsilon\right) + \underline{m}^2\left(1-\epsilon^{-1}\right) \end{aligned}$$

for every $\epsilon \in \mathbb{R}_*^+$ by using Cauchy's interpolated inequality, so that by choosing $\epsilon = \tan^2 \theta^*$ we obtain

$$d_{m,\lambda}^2 \geq \underline{m}^2 \left(1 - \tan^{-2} \theta^*\right) > 0$$

thanks to our choice of $\theta^*$. On the other hand, if $\operatorname{Re} \lambda \in [\underline{m}, +\infty)$ we get

$$d_{m,\lambda} \geq \operatorname{Im} \lambda \geq \operatorname{Re} \lambda \tan \theta^* \geq \underline{m} \tan \theta^* > 0.$$

The remaining cases when $\arg \lambda \leq -\theta^*$ with $\operatorname{Re} \lambda > 0$, $\operatorname{Im} \lambda < 0$, or when $\operatorname{Re} \lambda \leq 0$ can be dealt with in a similar way, thereby proving (84). ■

Finally, Hypothesis (D) is a straightforward consequence of (K′), (M′) and Gauss' divergence theorem if we choose, for instance, $\mathcal{D} = \mathcal{C}_0^2(D, \mathbb{C})$, the space of all complex-valued, twice continuously differentiable functions with compact support in $D$.

**Remarks.** (1) The statement of Proposition 2 is, therefore, a direct consequence of the above considerations and our main theorem since (K′) obviously implies (K) while (M′) implies (M). Indeed, by uniqueness, the evolution systems $U_A(t,s)_{0 \leq s \leq t \leq T}$ and $U_{A+B_\varepsilon}(t,s)_{0 \leq s \leq t \leq T}$ of Proposition 2 are then exactly the same as those defined by (9) and (17), respectively. But the natural question that is now emerging is whether the product formula (67) might hold under (K) and (M) alone; this is not immediate for Hypothesis (D) is *not* necessarily verified under these two conditions and, furthermore, some aspects of our proof of (32) are *not* completely independent of the existence proof for $U_{A+B}(t,s)_{0 \leq s \leq t \leq T}$. In fact, a rigorous proof of (67) under the sole set of conditions (K) and (M) is lacking at the moment, though we conjecture that this result is true. In any case, this brings us back to the third remark following the statement of the theorem.

(2) The fact that (67) holds with $U_{A+B_\varepsilon}(t,s)$ given by (17) where $G_{A+B_\varepsilon}$ is now the parabolic Green's function associated with the differential operator in (65) allows one to express the solution to this problem in the form of a Feynman-Kac formula. This is of course invaluable information for what regards the analysis of solutions to related semilinear initial-boundary value problems. However, we will not dwell on this any further in this paper, as we want to defer such detailed applications to a separate publication.

The above conjecture is all the more reinforced by the fact that some of the hypotheses of the preceding section are not necessary for our formulae to be valid in some simpler models. A case in point is the following example, which is a particular case of (65), namely, the class of parabolic initial-value problems of the form

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= k(t) \triangle u(x,t) - \varkappa u(x,t), \quad (x,t) \in D \times (0,T], \\ u(x,0) &= u_0(x), \quad x \in D, \\ \frac{\partial u(x,t)}{\partial n(x)} &= 0, \quad (x,t) \in \partial D \times (0,T], \end{aligned} \tag{85}$$

where $k \in \mathcal{C}^1([0,T], \mathbb{R}_*^+)$. Hypothesis (K) is here trivially satisfied and the self-adjoint, positive operator $A(t) := -k(t)\triangle + \varkappa$ in $L^2(D,\mathbb{C})$ is defined on the *time-independent* domain

$$\mathcal{D}(A(t)) = \left\{ v \in H^2(D,\mathbb{C}) : (\nabla v(x), n(x))_{\mathbb{C}^d} = 0,\ x \in \partial D \right\}$$

since $k$ is a scalar function. Furthermore, Hypothesis (A1) holds if $\varkappa$ is sufficiently large but Hypothesis (A2) does *not* since we cannot expect (22) to be satisfied without requiring the derivative $k'$ to be Hölder continuous. Nevertheless, there exists an evolution system $U_A(t,s)_{0\leq s\leq t\leq T}$ for (85), namely,

$$U_A(t,s) = e^{-\varkappa(t-s)} \exp\left[\int_s^t dy k(y)\triangle\right], \tag{86}$$

and our point with this example is to show that we can also reconstruct (86) by means of (33). In fact, on the one hand we have

$$\begin{aligned} &\prod_{\gamma=n-1}^{0} \exp\left[-\frac{t-s}{n} A\left(s + \frac{\gamma}{n}(t-s)\right)\right] \\ = \quad & e^{-\varkappa(t-s)} \exp\left[\frac{t-s}{n} \sum_{\gamma=0}^{n-1} k\left(s + \frac{\gamma}{n}(t-s)\right)\triangle\right], \end{aligned} \tag{87}$$

and on the other hand we may write

$$\begin{aligned} &\sum_{\gamma=0}^{n-1} k\left(s + \frac{\gamma}{n}(t-s)\right) \\ = \quad & \int_0^{n-1} dy k\left(s + \frac{y}{n}(t-s)\right) + \frac{1}{2}\left(k(s) + k\left(s + \frac{n-1}{n}(t-s)\right)\right) \\ & + \frac{t-s}{n}\int_0^{n-1} dy \left(y - [y] - \frac{1}{2}\right) k'\left(s + \frac{y}{n}(t-s)\right) \end{aligned} \tag{88}$$

by Euler's summation formula, with $[y]$ the integral part of $y$ (see, for instance, [17]). Regarding the first term on the right-hand side of (88) we have

$$\lim_{n\to+\infty} \frac{t-s}{n} \int_0^{n-1} dy k\left(s + \frac{y}{n}(t-s)\right) = \int_s^t dy k(y) \tag{89}$$

since $k$ is smooth, while for the remaining two terms we get

$$\begin{aligned} &\lim_{n\to+\infty} \frac{t-s}{2n}\left(k(s) + k\left(s + \frac{n-1}{n}(t-s)\right)\right) \\ = \quad & \lim_{n\to+\infty}\left(\frac{t-s}{n}\right)^2 \int_0^{n-1} dy \left(y - [y] - \frac{1}{2}\right) k'\left(s + \frac{y}{n}(t-s)\right) = 0, \end{aligned} \tag{90}$$

the last equality in the preceding expression being a consequence of the boundedness of $y \mapsto y - [y] - \frac{1}{2}$ and $k'$. Consequently, owing to (87)-(90) and to the

$\mathcal{C}_0$-continuity of the underlying diffusion semigroup generated by the Laplacian we get

$$\exp\left[\int_s^t dyk(y)\triangle\right] = \lim_{n\to+\infty} \exp\left[\frac{t-s}{n}\sum_{\gamma=0}^{n-1} k\left(s+\frac{\gamma}{n}(t-s)\right)\triangle\right] \tag{91}$$

in the strong topology of $\mathcal{L}(L^2(D,\mathbb{C})$, as desired.

Along with (91), we remark that in the preceding example Hypothesis (D) also holds if we choose once more $\mathcal{D}= \mathcal{C}_0^2(D,\mathbb{C})$. Since that hypothesis plays an important rôle in the proof of (32) and (33) within our abstract setting, we may then tend to believe that it is also necessary for those product formulae to hold. We now show that even this is *not* the case by considering a third example related to one very briefly mentioned at the end of [20]. Let us consider the initial value problem

$$\begin{aligned} \frac{du(x,t)}{dt} &= -\frac{u(x,t)}{(t-x)^2}, \quad (x,t)\in(0,1)\times(0,1]\,, \\ u(x,0) &= u_0, \quad x\in(0,1) \end{aligned} \tag{92}$$

in $L^2((0,1),\mathbb{C})$, that is, (21) with $T=1$, $B(t)=0$ and the $A(t)$'s self-adjoint, multiplication operators defined by

$$A(t)v(x) := \frac{v(x)}{(t-x)^2} \tag{93}$$

on the maximal, time-dependent domains

$$\mathcal{D}(A(t)) = \left\{v\in L^2((0,1),\mathbb{C}) : \int_0^1 dx\frac{|v(x)|^2}{(t-x)^4} < +\infty\right\} \tag{94}$$

where $t\in[0,1]$. In this case Hypothesis (D) does *not* hold since we have the rather extreme opposite situation where

$$\cap_{t\in[0,1]}\mathcal{D}(A(t)) = \{0\}\,. \tag{95}$$

In fact, let $v\in\cap_{t\in[0,1]}\mathcal{D}(A(t))$ and $\epsilon\in\mathbb{R}_*^+$ sufficiently small; on the one hand, by using Schwarz inequality we have

$$\begin{aligned} &(2\epsilon)^{-1}\int_{t-\epsilon}^{t+\epsilon} dx\,|v(x)| \\ \leq\ &(2\epsilon)^{-1}\left(\int_{t-\epsilon}^{t+\epsilon} dx(t-x)^4\right)^{\frac{1}{2}} \|A(t)v\|_2 \leq c\epsilon^{\frac{3}{2}}\|A(t)v\|_2 \to 0 \end{aligned} \tag{96}$$

for *every* $t\in(0,1)$ as $\epsilon\to 0_+$. On the other hand, we infer from standard one-dimensional Lebesgue integration theory that

$$\lim_{\epsilon\to 0_+}(2\epsilon)^{-1}\int_{t-\epsilon}^{t+\epsilon} dx\,|v(x)| = |v(t)|$$

for almost every $t \in (0,1)$, which, together with (96), indeed implies $v = 0$ in $L^2((0,1),\mathbb{C})$.

In spite of this fact and by means of yet another application of Euler's summation formula, we now prove that (33) holds for all $s,t$ with $0 \leq s \leq t \leq 1$ in the strong topology of $L^2((0,1),\mathbb{C})$, thereby showing that the reconstruction of the full evolution system from the semigroups generated by the $A(t)$'s is possible in this case as well. On the one hand, the holomorphic semigroup generated by $-A(t)$ is the contraction semigroup given by

$$\exp\left[-sA(t)\right]v(x) = \exp\left[-\frac{s}{(t-x)^2}\right]v(x) \tag{97}$$

for every $s \in \mathbb{R}^+$and any $v \in L^2((0,1),\mathbb{C})$. On the other hand, an explicit calculation from (92) shows that the corresponding evolution system $U_A(t,s)_{0\leq s\leq t\leq 1}$ in $\mathcal{B} = L^2((0,1),\mathbb{C})$ also exists in the form of the multiplication operators

$$U_A(t,s)v(x) = \left\{ \begin{array}{c} \exp\left[(t-x)^{-1} - (s-x)^{-1}\right]v(x) \text{ if } \ x \in (0,s) \cup (t,1)\,, \\ 0 \text{ if } \ x \in (s,t)\,. \end{array} \right\} \tag{98}$$

We begin our analysis of the reconstruction with the following auxiliary result.

**Proposition 3.** *For every $v \in L^2((0,1),\mathbb{C})$ and all $s,t$ with $0 \leq s < t \leq 1$ we have*

$$\lim_{n\to+\infty}\int_s^t dx \exp\left[-2n(t-s)\sum_{\gamma=0}^{n-1}\frac{1}{(\gamma(t-s)-n(x-s))^2}\right]|v(x)|^2 = 0.$$

**Proof.** It is sufficient to prove that

$$\begin{aligned} &\int_s^t dx \exp\left[-2n(t-s)\sum_{\gamma=0}^{n-1}\frac{1}{(\gamma(t-s)-n(x-s))^2}\right]|v(x)|^2 \\ \leq \ &\exp\left[-\frac{4n}{t-s}\right]\|v\|_2^2\,. \end{aligned} \tag{99}$$

In order to achieve this we write

$$\begin{aligned} &\int_s^t dx \exp\left[-2n(t-s)\sum_{\gamma=0}^{n-1}\frac{1}{(\gamma(t-s)-n(x-s))^2}\right]|v(x)|^2 \qquad (100) \\ = \ &\sum_{\delta=0}^{n-1}\int_{s+\delta\frac{(t-s)}{n}}^{s+(\delta+1)\frac{(t-s)}{n}} dx \exp\left[-2n(t-s)\sum_{\gamma=0}^{n-1}\frac{1}{(\gamma(t-s)-n(x-s))^2}\right]|v(x)|^2 \end{aligned}$$

and observe that the inequalities

$$(\gamma-\delta-1)\,(t-s) < \gamma(t-s) - n(x-s) < (\gamma-\delta)\,(t-s)$$

hold for every $x$ and every $\gamma,\delta\in\{0,...,n-1\}$ in each of the integrals on the right-hand side of (100). Consequently, if $\gamma\leq\delta$ we get the lower bound

$$\frac{1}{(\gamma(t-s)-n(x-s))^2}>\frac{1}{(\gamma-\delta-1)^2(t-s)^2}$$

while if $\gamma\geq\delta+1$ we have

$$\frac{1}{(\gamma(t-s)-n(x-s))^2}>\frac{1}{(\gamma-\delta)^2(t-s)^2}.$$

Therefore we obtain the estimate

$$\begin{aligned}
&\sum_{\gamma=0}^{n-1}\frac{1}{(\gamma(t-s)-n(x-s))^2}\\
=\;&\sum_{\gamma=0}^{\delta}\frac{1}{(\gamma(t-s)-n(x-s))^2}+\sum_{\gamma=\delta+1}^{n-1}\frac{1}{(\gamma(t-s)-n(x-s))^2}\\
>\;&\frac{1}{(t-s)^2}\left(\sum_{\gamma=0}^{\delta}\frac{1}{(\gamma-\delta-1)^2}+\sum_{\gamma=\delta+1}^{n-1}\frac{1}{(\gamma-\delta)^2}\right)>\frac{2}{(t-s)^2}
\end{aligned}$$

uniformly in $x$, $\delta$ and $n$, so that the substitution of the preceding inequality into the right-hand side of (100) indeed leads to

$$\begin{aligned}
&\int_s^t dx\exp\left[-2n(t-s)\sum_{\gamma=0}^{n-1}\frac{1}{(\gamma(t-s)-n(x-s))^2}\right]|v(x)|^2\\
\leq\;&\exp\left[-\frac{4n}{t-s}\right]\sum_{\delta=0}^{n-1}\int_{s+\delta\frac{(t-s)}{n}}^{s+(\delta+1)\frac{(t-s)}{n}}dx\,|v(x)|^2\leq\exp\left[-\frac{4n}{t-s}\right]\|v\|_2^2\,,
\end{aligned}$$

which is (99). ■

It is more complicated to get the relevant estimates when $x\in(0,s)\cup(t,1)$. To this end let us introduce the functions $f_{n,t,s}(.,x):[0,n-1]\mapsto\mathbb{R}_*^+$ defined by

$$f_{n,t,s}(y,x):=\frac{1}{(y(t-s)-n(x-s))^2}\tag{101}$$

for every $n\in\mathbb{N}^+\cap[2,+\infty)$, along with the function $f_{t,s}:(0,s)\cup(t,1)\mapsto\mathbb{R}_*^-$ given by

$$f_{t,s}(x):=(t-x)^{-1}-(s-x)^{-1}.\tag{102}$$

Our second auxiliary result is the following.

**Proposition 4.** *For every $v\in L^2((0,1),\mathbb{C})$ and all $s,t$ with $0\leq s<t\leq1$ we have*

$$\lim_{n\to+\infty}\int_0^s dx\left|\exp\left[-n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]-\exp\left[f_{t,s}(x)\right]\right|^2|v(x)|^2=0$$

*and*

$$\lim_{n\to+\infty}\int_t^1 dx\left|\exp\left[-n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]-\exp\left[f_{t,s}(x)\right]\right|^2|v(x)|^2=0.$$

The proof of this proposition rests on one crucial lemma. We first remark that the $f_{n,t,s}(.,x)$'s are well-defined and continuously differentiable on $[0,n-1]$ for every $x\in(0,s)\cup(t,1)$ since their denominators do not vanish there. Consequently, we may write

$$\begin{aligned}
&\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\\
=\;&\int_0^{n-1}dyf_{n,t,s}(y,x)+\frac{1}{2}\left(f_{n,t,s}(0,x)+f_{n,t,s}(n-1,x)\right)\\
&-\int_0^{n-1}dy\psi(y)f''_{n,t,s}(y,x)
\end{aligned}\tag{103}$$

as a variant of Euler's summation formula, where we have defined

$$\psi(y):=\int_0^y dz\left(z-[z]-\frac{1}{2}\right).$$

We remark that $\psi$ satisfies the inequalities

$$-\frac{1}{8}\leq\psi(y)\leq 0\tag{104}$$

for every $y\in\mathbb{R}$; regarding (101) we then have the following.

**Lemma 9.** *For every $x\in(0,s)\cup(t,1)$ and all $s,t$ with $0\leq s<t\leq 1$ we have*

$$\begin{aligned}
&\exp\left[-n(t-s)\int_0^{n-1}dyf_{n,t,s}(y,x)\right]\\
=\;&\exp\left[\left(t-x-\frac{t-s}{n}\right)^{-1}-(s-x)^{-1}\right]
\end{aligned}\tag{105}$$

*along with*

$$\begin{aligned}
&\exp\left[-\frac{n(t-s)}{2}\left(f_{n,t,s}(0,x)+f_{n,t,s}(n-1,x)\right)\right]\\
=\;&\exp\left[-\frac{t-s}{2}\left(\frac{1}{n(x-s)^2}+\frac{n}{(n(t-x)-(t-s))^2}\right)\right].
\end{aligned}\tag{106}$$

*Moreover, we have the estimate*

$$\begin{aligned}
& \exp\left[\frac{(t-s)^2}{4}\left(\frac{n}{(n(t-x)-(t-s))^3}+\frac{1}{n^2(x-s)^3}\right)\right] \\
\leq\ & \exp\left[n(t-s)\int_0^{n-1} dy\psi(y)f''_{n,t,s}(y,x)\right] \leq 1. \qquad (107)
\end{aligned}$$

**Proof.** Relations (105) and (106) follow from an explicit evaluation of the first two terms on the right-hand side of (103). Furthermore, from (104) we have

$$\begin{aligned}
& -\frac{n(t-s)}{8}\left(f'_{n,t,s}(n-1,x)-f'_{n,t,s}(0,x)\right) \\
=\ & -\frac{n(t-s)}{8}\int_0^{n-1} dy f''_{n,t,s}(y,x) \\
\leq\ & n(t-s)\int_0^{n-1} dy\psi(y)f''_{n,t,s}(y,x) \leq 0
\end{aligned}$$

since $f_{n,t,s}(.,x)$ is convex, from which (107) follows immediately. ∎

For the sake of simplicity we now introduce a shorthand notation for all three exponential arguments above, namely,

$$\Theta_{n,t,s}(x) := n(t-s)\int_0^{n-1} dy f_{n,t,s}(y,x), \qquad (108)$$

$$\Phi_{n,t,s}(x) := \frac{n(t-s)}{2}\left(f_{n,t,s}(0,x)+f_{n,t,s}(n-1,x)\right) \qquad (109)$$

and

$$\Psi_{n,t,s}(x) := n(t-s)\int_0^{n-1} dy\psi(y)f''_{n,t,s}(y,x). \qquad (110)$$

We then have the following.

**Proof of Proposition 4.** For every $x \in (0,s)\cup(t,1)$ and all $s,t$ with $0 \leq s < t \leq 1$ we may write

$$\begin{aligned}
& \exp\left[-n(t-s)\sum_{\gamma=0}^{n-1} f_{n,t,s}(\gamma,x)\right] - \exp\left[f_{t,s}(x)\right] \\
=\ & \exp\left[-\Phi_{n,t,s}(x)\right]\exp\left[\Psi_{n,t,s}(x)\right]\left(\exp\left[-\Theta_{n,t,s}(x)\right]-\exp\left[f_{t,s}(x)\right]\right) \\
& +\left(\exp\left[-\Phi_{n,t,s}(x)\right]\exp\left[\Psi_{n,t,s}(x)\right]-1\right)\exp\left[f_{t,s}(x)\right]
\end{aligned}$$

as a consequence of (103) and (108)-(110); moreover, from (102), (106) and (107) we have

$$\exp\left[f_{t,s}(x)\right] \leq 1,$$

$$\exp\left[-\Phi_{n,t,s}(x)\right]\exp\left[\Psi_{n,t,s}(x)\right]\leq 1$$

and

$$\lim_{n\to+\infty}\exp\left[-\Phi_{n,t,s}(x)\right]\exp\left[\Psi_{n,t,s}(x)\right]=1, \tag{111}$$

respectively. For every $v\in L^2((0,1),\mathbb{C})$, almost every $x\in(0,s)\cup(t,1)$ and all $s,t$ with $0\leq s<t\leq 1$ we then get

$$\begin{aligned}
&\left|\exp\left[-n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]-\exp\left[f_{t,s}(x)\right]\right|^2|v(x)|^2\\
\leq\ & 2\Big(\left|\exp\left[-\Theta_{n,t,s}(x)\right]-\exp\left[f_{t,s}(x)\right]\right|^2|v(x)|^2\\
&+\left|\exp\left[-\Phi_{n,t,s}(x)\right]\exp\left[\Psi_{n,t,s}(x)\right]-1\right|^2|v(x)|^2\Big),
\end{aligned}$$

so that by taking (105) and (111) into account we obtain

$$\lim_{n\mapsto+\infty}\left|\exp\left[-n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]-\exp\left[f_{t,s}(x)\right]\right|^2|v(x)|^2=0.$$

The result then follows from a simple dominated convergence argument. ■

It is now plain that (33) emerges as a consequence of Propositions 3 and 4; in fact, thanks to (97) and (101) we have

$$\begin{aligned}
&\prod_{\gamma=n-1}^{0}\exp\left[-\frac{t-s}{n}A\left(s+\frac{\gamma}{n}(t-s)\right)\right]v(x)\\
=\ &\exp\left[-n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]v(x)
\end{aligned}$$

for every $v\in L^2((0,1),\mathbb{C})$, almost every $x\in(0,1)$ and all $s,t$ with $0\leq s<t\leq 1$. Therefore, from (98), (102) along with Propositions 3 and 4 we indeed get

$$\begin{aligned}
&\left\|U_A(t,s)v-\prod_{\gamma=n-1}^{0}\exp\left[-\frac{t-s}{n}A\left(s+\frac{\gamma}{n}(t-s)\right)\right]v\right\|_2^2\\
=\ &\int_0^s dx\left|\exp\left[-n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]-\exp\left[f_{t,s}(x)\right]\right|^2|v(x)|^2\\
&+\int_s^t dx\exp\left[-2n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]|v(x)|^2\\
&+\int_t^1 dx\left|\exp\left[-n(t-s)\sum_{\gamma=0}^{n-1}f_{n,t,s}(\gamma,x)\right]-\exp\left[f_{t,s}(x)\right]\right|^2|v(x)|^2\to 0
\end{aligned}$$

as $n \to +\infty$. ■

Our last example is motivated by some questions related to the theme of singular perturbations of self-adjoint operators. It also illustrates the fact that the theory we developed in the preceding section can be applied to evolution problems defined on *unbounded* domains of Euclidean space and, of course, to the case of *unbounded* $B(t)$'s. Thus, let us consider the parabolic initial-value problem

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \left( \frac{d^2}{dx^2} - \mathrm{s}(t)\delta_{x^*} - \varkappa - \varepsilon m(x,t) \right) u(x,t), \quad (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{112}$$

corresponding to a time-dependent, zero-range perturbation at $x^* \in \mathbb{R}$ involving Dirac's distribution $\delta_{x^*}$, with $u_0 \in L^2(\mathbb{R},\mathbb{R})$ and $\varkappa, \varepsilon \in \mathbb{R}^+$ parameters as before. Problems such as (112) with one or several perturbations supported on a discrete set of points in one or several space dimensions may play an important rôle in the mathematical analysis of the dynamics of one particle diffusing through a set of very small obstacles varying with time (see, for instance, [11] for further information on the subject).

Regarding the strength of the zero-range perturbation we introduce the following condition:

(S) The function $\mathrm{s} : [0,T] \mapsto \mathbb{R}^+$ is differentiable, and its derivative $\mathrm{s}'$ is Hölder continuous with Hölder exponent $\sigma' \in (0,1]$.

As for the lower order term we impose the following hypothesis:

(M$''$) The function $m : \mathbb{R} \times [0,T] \mapsto \mathbb{R}^+$ is measurable with

$$x \mapsto \mathsf{M}(x) := \sup_{t \in [0,T]} m(x,t) \in L^2(\mathbb{R},\mathbb{R}^+). \tag{113}$$

Furthermore, the function $t \mapsto m(x,t)$ is differentiable on $[0,T]$ for every $x \in \mathbb{R}$ and there exist a constant $c'_* \in \mathbb{R}^+_*$, a function $\mathsf{H} \in L^2(\mathbb{R},\mathbb{R}^+)$ such that the Hölder continuity estimate

$$\left| \frac{\partial m(x,t)}{\partial t} - \frac{\partial m(x,s)}{\partial s} \right| \leq c'_* \mathsf{H}(x) \, |t-s|^{\sigma'} \tag{114}$$

holds for every $x \in \mathbb{R}$ and every $s, t \in [0,T]$, where $\sigma' \in (0,1]$ may be chosen to be the same as in Hypothesis (S). Finally, we have

$$x \mapsto \mathsf{N}(x) := \sup_{t \in [0,T]} \left| \frac{\partial m(x,t)}{\partial t} \right| \in L^2(\mathbb{R},\mathbb{R}^+). \tag{115}$$

As above, it is here also possible to construct a self-adjoint, positive realization of the operator

$$A(t) := -\frac{d^2}{dx^2} + \mathrm{s}(t)\delta_{x^*} + \varkappa \tag{116}$$

in $L^2(\mathbb{R},\mathbb{C})$, this time as a form sum by considering the Hermitian sesquilinear form $\mathsf{a}:[0,T]\times H^1(\mathbb{R},\mathbb{C})\times H^1(\mathbb{R},\mathbb{C})\mapsto\mathbb{C}$ defined by

$$\mathsf{a}(t,v,w):=\int_{\mathbb{R}}dxv'(x)\overline{w}'(x)+\mathrm{s}(t)v(x^*)\overline{w}(x^*)+\varkappa\int_{\mathbb{R}}dxv(x)\overline{w}(x). \tag{117}$$

In this case, the corresponding *time-dependent* domain for (116) is given by

$$\mathcal{D}(A(t))=\left\{v\in H^1(\mathbb{R},\mathbb{C})\cap H^2(\mathbb{R}\diagdown\{x^*\},\mathbb{C}):v'(x^*_+)-v'(x^*_-)=\mathrm{s}(t)v(x^*)\right\}$$

for every $t\in[0,T]$, where

$$v'(x^*_+):=\lim_{y\searrow 0}v'(x^*+y)$$

and

$$v'(x^*_-):=\lim_{y\searrow 0}v'(x^*-y)$$

(see, for instance, [3]).

From Hypothesis (S) and standard one-dimensional Sobolev theory, it follows that (117) satisfies estimates similar to (5) and (6). Consequently, since $\mathrm{s}(t)\geq 0$ for every $t\in[0,T]$ we infer from the general theory of [35] that $-A(t)$ generates a holomorphic semigroup of contractions $\exp\left[-sA(t)\right]_{s\geq 0}$ in $L^2(\mathbb{R},\mathbb{C})$, which means that Hypotheses (A1) and (A4) hold provided we choose again $\varkappa$ sufficiently large, for instance $\varkappa\geq 1$. Furthermore, $-A(t)$ also generates an evolution system $U_A(t,s)_{0\leq s\leq t\leq T}$ in $L^2(\mathbb{R},\mathbb{C})$.

Therefore, in order to prove (32) for (112) we can begin by verifying (A2), (A3) and for this we wish to exploit the fact that the resolvent operator for (116) is known quite explicitly, rather than rely on the variational structure of the problem. More precisely, for every $\lambda\in\mathbb{C}\diagdown\mathbb{R}^+$ we write $\mathsf{k}^2:=\varkappa-\lambda$ with $\operatorname{Re}\mathsf{k}>0$ and then have by Krein's formula for resolvents (see, for instance, [3])

$$\begin{aligned}&R(A(t),\lambda)f(x)\\=&\left(-\frac{d^2}{dx^2}+\mathrm{s}(t)\delta_{x^*}+\mathsf{k}^2\right)^{-1}f(x)\\=&\frac{1}{2\mathsf{k}}\int_{\mathbb{R}}dye^{-\mathsf{k}|x-y|}f(y)-\frac{\mathrm{s}(t)}{2\mathsf{k}\left(\mathrm{s}(t)+2\mathsf{k}\right)}\left(\int_{\mathbb{R}}dye^{-\mathsf{k}|x^*-y|}f(y)\right)e^{-\mathsf{k}|x-x^*|}\end{aligned} \tag{118}$$

for every $f\in L^2(\mathbb{R},\mathbb{C})$ and almost every $x\in\mathbb{R}$, from which we obtain

$$\frac{\partial}{\partial t}R(A(t),\lambda)f(x)=-\frac{\mathrm{s}'(t)}{\left(\mathrm{s}(t)+2\mathsf{k}\right)^2}\left(\int_{\mathbb{R}}dye^{-\mathsf{k}|x^*-y|}f(y)\right)e^{-\mathsf{k}|x-x^*|} \tag{119}$$

thanks to the differentiability of s. From this we first get the following result.

**Lemma 10.** *Assume that Hypothesis (S) holds; then, there exists a constant $a_2\in\mathbb{R}^+_*$ such that the Hölder continuity estimate*

$$\left\|\frac{d}{dt}A^{-1}(t)f-\frac{d}{ds}A^{-1}(s)f\right\|_2\leq a_2\left|t-s\right|^{\sigma'}\left\|f\right\|_2 \tag{120}$$

*holds for all $s,t\in[0,T]$ and every $f\in L^2(\mathbb{R},\mathbb{C})$, with $\sigma'$ the Hölder exponent in (S).*

**Proof.** Relation (119) with $\lambda=0$ reduces to

$$\frac{d}{dt}A^{-1}(t)f(x)=-\frac{\mathrm{s}'(t)}{\left(\mathrm{s}(t)+2\sqrt{\varkappa}\right)^2}\left(\int_{\mathbb{R}}dye^{-\sqrt{\varkappa}|x^*-y|}f(y)\right)e^{-\sqrt{\varkappa}|x-x^*|},\quad(121)$$

and furthermore we infer from Hypothesis (S) that the function

$$t\mapsto\frac{\mathrm{s}'(t)}{\left(\mathrm{s}(t)+2\sqrt{\varkappa}\right)^2}$$

is Hölder continuous on $[0,T]$ with Hölder exponent $\sigma'$. Moreover, we can estimate the integral in (121) by means of Schwarz inequality, so that we eventually get

$$\begin{aligned}&\left\|\frac{d}{dt}A^{-1}(t)f-\frac{d}{ds}A^{-1}(s)f\right\|_2\\ \leq\ & c\left|t-s\right|^{\sigma'}\left(\int_{\mathbb{R}}dxe^{-2\sqrt{\varkappa}|x|}\right)^{\frac{1}{2}}\|f\|_2\\ \leq\ & c\left|t-s\right|^{\sigma'}\|f\|_2\end{aligned}$$

for every $s,t\in[0,T]$ and every $f\in L^2(\mathbb{R},\mathbb{C})$. ■

While it is plain that (120) leads to Hypothesis (A2), we now turn to the verification of (A3). For this we have the following result.

**Lemma 11.** *Assume that Hypothesis (S) holds; then, there exists a constant $a_3\in\mathbb{R}_*^+$ such that the estimate*

$$\left\|\frac{\partial}{\partial t}R(A(t),\lambda)f\right\|_2\leq a_3\left|\lambda\right|^{-1}\|f\|_2\quad(122)$$

*holds for every $t\in[0,T]$, any $f\in L^2(\mathbb{R},\mathbb{C})$ and each $\lambda\in\mathcal{S}_\theta\diagdown\{0\}$.*

**Proof.** From (119) we easily obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}R(A(t),\lambda)f\right\|_2\\ \leq\ & \frac{c}{\left|\mathrm{s}(t)+2\mathsf{k}\right|^2}\left(\int_{\mathbb{R}}dye^{-2\operatorname{Re}\mathsf{k}|y|}\right)\|f\|_2\\ \leq\ & \frac{c}{\left|\mathsf{k}\right|^2\operatorname{Re}\mathsf{k}}\|f\|_2\end{aligned}\quad(123)$$

for every $f\in L^2(\mathbb{R},\mathbb{C})$ and every $\lambda\in\mathbb{C}\diagdown\mathbb{R}^+$, where the last inequality follows from an explicit evaluation of the integral and the fact that $\mathrm{s}\geq0$, $\operatorname{Re}\mathsf{k}>0$.

Without restricting the generality we now take $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and first prove the existence of a constant $c_\theta \in \mathbb{R}_*^+$ such that the inequality

$$|\mathsf{k}|^2 \geq c_\theta \left(1 + |\lambda|\right) \tag{124}$$

holds for every $\lambda \in \mathcal{S}_\theta \diagdown \{0\}$; this is obvious with a constant independent of $\theta$ if $\operatorname{Re}\lambda \leq 0$ (with $\lambda \neq 0$ when $\operatorname{Re}\lambda = 0$) since

$$\left|\mathsf{k}^2\right|^2 = \varkappa^2 - 2\varkappa \operatorname{Re}\lambda + |\lambda|^2 .$$

Furthermore, if $\arg\lambda \geq \theta$ with $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda > 0$, or if $\arg\lambda \leq -\theta$ with $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda < 0$ we get from the preceding relation and Cauchy's interpolated inequality the estimate

$$\left|\mathsf{k}^2\right|^2 \geq \left(1 - \epsilon^{-1}\right)\varkappa^2 + \left(1 - \epsilon \tan^{-2}\theta\right)|\lambda|^2 \tag{125}$$

for every $\epsilon \in \mathbb{R}_*^+$, so that by choosing for instance $\epsilon = \frac{1}{2}\left(1 + \tan^2\theta\right)$ we can make the two terms in (125) positive, which indeed leads to (124).

We proceed by proving that

$$\inf_{\lambda \in \mathcal{S}_\theta \diagdown \{0\}} \operatorname{Re}\mathsf{k} > 0. \tag{126}$$

We have

$$\operatorname{Re}\mathsf{k} = \left(\frac{\operatorname{Re}\mathsf{k}^2 + |\mathsf{k}|^2}{2}\right)^{\frac{1}{2}} \tag{127}$$

so that if $\operatorname{Re}\lambda \leq \varkappa$ we get $\operatorname{Re}\mathsf{k}^2 \geq 0$ and hence

$$\operatorname{Re}\mathsf{k} \geq c_\theta \tag{128}$$

from (124) and (127). In order to get a similar bound for the case $\operatorname{Re}\lambda > \varkappa$, it is sufficient to prove that $\operatorname{Re}\mathsf{k}$ is bounded from below by a function of $\operatorname{Re}\lambda$ having a positive minimum at $\operatorname{Re}\lambda = \varkappa$; to this end, let us define $\mathsf{F}_\theta$:$[\varkappa, +\infty) \mapsto \mathbb{R}_*^+$ by

$$\mathsf{F}_\theta(\xi) := \frac{\varkappa - \xi + \left(\varkappa^2 - 2\varkappa\xi + \xi^2(1 + \tan^2\theta)\right)^{\frac{1}{2}}}{2}. \tag{129}$$

A direct calculation shows that $\mathsf{F}_\theta$ is monotone increasing, so that the comparison of (127) and (129) gives

$$\left(\operatorname{Re}\mathsf{k}\right)^2 \geq \mathsf{F}_\theta(\operatorname{Re}\lambda) \geq \mathsf{F}_\theta(\varkappa) \geq \frac{\tan\theta}{2} > 0$$

and thereby indeed a bound identical to (128). Therefore (126) holds, which, together with (123) and (124), gives (122). ■

We proceed with the verification of (B1)-(B4). The multiplication operators $B_\varepsilon(t) := \varepsilon B(t)$ defined by

$$B(t)v := m(.,t)v \tag{130}$$

are in this case self-adjoint and positive on the maximal, time-dependent domains

$$\mathcal{D}(B(t)) = \left\{ v \in L^2(\mathbb{R},\mathbb{C}) : \int_{\mathbb{R}} dx \, |m(x,t)v(x)|^2 < +\infty \right\}$$

for every $t \in [0,T]$ and, although the $B(t)$'s are in general unbounded, the crucial fact that entails the validity of (B1) is the boundedness of the operators $B(t)R(A(t),\lambda)$ on $L^2(\mathbb{R},\mathbb{C})$. More precisely we have the following result.

**Lemma 12.** *Assume that Hypothesis (S) and (113) hold; then, there exists a constant $c_\theta \in \mathbb{R}_*^+$ such that the inequality*

$$\|B(t)R(A(t),\lambda)f\|_2 \leq c_\theta \|\mathsf{M}\|_2 \|f\|_2 \tag{131}$$

*is valid for every $t \in [0,T]$, any $f \in L^2(\mathbb{R},\mathbb{C})$ and each $\lambda \in \mathcal{S}_\theta$. Thus, the $B_\varepsilon(t)$'s satisfy (B1) for every $\varepsilon \in \mathbb{R}^+$ sufficiently small.*

**Proof.** From (118), (130) and estimates similar to those carried out in the proofs of the last two lemmas we easily obtain

$$\begin{aligned} &\|B(t)R(A(t),\lambda)f\|_2^2 \\ &\leq \frac{c}{|\mathsf{k}|^4 \operatorname{Re} \mathsf{k}} \left(1 + |\mathsf{k}|^2\right) \left( \int_{\mathbb{R}} dx m^2(x,t) \right) \|f\|_2^2 , \end{aligned}$$

of which (131) is a consequence because of (113), (124) and (128). The remaining statement of the lemma is then immediate by setting $v = R(A(t),\lambda)f$ in (131) for every $v \in \mathcal{D}(A(t))$. ■

Next, we have the following result whose proof is relatively long but similar to the last three and therefore omitted.

**Lemma 13.** *Assume that Hypothesis (S) and (113)-(115) hold; then, there exists a constant $c_* \in \mathbb{R}_*^+$ depending on $\theta$, $\|\mathsf{M}\|_2$, $\|\mathsf{H}\|_2$ and $\|\mathsf{N}\|_2$ such that the Hölder continuity estimate*

$$\left\| \frac{d}{dt} B(t) A^{-1}(t) f - \frac{d}{ds} B(s) A^{-1}(s) f \right\|_2 \leq \; c_* \, |t-s|^{\sigma'} \|f\|_2$$

*is valid for all $s,t \in [0,T]$ and every $f \in L^2(\mathbb{R},\mathbb{C})$. Moreover, the function $t \mapsto \left\| \frac{\partial}{\partial t}(B(t)R(A(t),\lambda)) \right\|_\infty$ is continuously differentiable on $[0,T]$ with respect to the norm-topology of $\mathcal{L}(L^2(\mathbb{R},\mathbb{C}))$ and there exists a constant $c_\theta \in \mathbb{R}_*^+$ such that the inequality*

$$\left\| \frac{\partial}{\partial t} B(t) R(A(t),\lambda) f \right\|_2 \leq c_\theta \left( \|\mathsf{M}\|_2 + \|\mathsf{N}\|_2 \right) \|f\|_2$$

*holds for every $t \in [0,T]$, any $f \in L^2(\mathbb{R},\mathbb{C})$ and each $\lambda \in \mathcal{S}_\theta$. Thus, the $B_\varepsilon(t)$'s satisfy (B2) and (B3) for every $\varepsilon \in \mathbb{R}^+$.*

As for the verification of (B4), we can either proceed as in Lemma 8 or prove (58) directly by observing that

$$\exp\left[-(r-s)B(r)\right]f = \exp\left[-(r-s)m(.,r)\right]f$$

in $L^2(\mathbb{R},\mathbb{C})$. Then, for any $r \in [s,t]$ and any sequence $(r_n)_{n\in\mathbb{N}^+} \subset [s,t]$ such that $r_n \to r$ we have

$$\exp\left[-(r_n-s)m(x,r_n)\right]f(x) \to \exp\left[-(r-s)m(x,r)\right]f(x)$$

for almost every $x \in \mathbb{R}$ when $n \to +\infty$, as well as

$$\begin{aligned} & \left|\exp\left[-(r_n-s)m(x,r_n)\right]f(x) - \exp\left[-(r-s)m(x,r)\right]f(x)\right|^2 \\ \leq \quad & c\left|f(x)\right|^2 \end{aligned}$$

uniformly in $n$. Therefore, we get

$$\lim_{n\to+\infty} \exp\left[-(r_n-s)B(r_n)\right]f = \exp\left[-(r-s)B(r)\right]f$$

strongly in $L^2(\mathbb{R},\mathbb{C})$ by dominated convergence, which is the desired property.

Finally, Hypothesis (D) can be verified with

$$\mathcal{D} = \left\{v \in \mathcal{C}_0^2(\mathbb{R}\diagdown\{x^*\},\mathbb{C}) : v(x^*) = 0\right\} \tag{132}$$

which is dense in $L^2(\mathbb{R},\mathbb{C})$; indeed (27) and (29) trivially hold, as does (28) since the restriction of (116) to the domain (132) coincides with the *time-independent* operator $-\frac{d^2}{dx^2} + \varkappa$ (see, for instance, [3] or [11]).

The preceding considerations thus lead to the following result.

**Proposition 5.** *Assume that Hypotheses (S) and (M$''$) hold; then, all the conclusions of the theorem are valid for (112) for every $\varepsilon \in \mathbb{R}^+$ sufficiently small. In particular, the Trotter-Kato formula (32) and the reconstruction formula (33) hold in the strong topology of $\mathcal{L}(L^2(\mathbb{R},\mathbb{C}))$.*

**Remark.** The preceding example shows that in the particular case where the $B(t)$'s are self-adjoint multiplication operators on a Hilbert space, there is a much more direct way of proving the strong continuity of $r \mapsto \exp\left[-(r-s)B(r)\right]$ than that stemming from Hypothesis (B4), as it is sufficient to invoke the spectral theorem. However, in the general case the full force of (B4) is indeed deemed appropriate according to the proof of Lemma 5.

We conclude this article by establishing a connection between the above theory and the corresponding evolution problems for Schrödinger-type equations of quantum mechanics, namely,

$$\begin{aligned} i\frac{du(t)}{dt} &= (A(t)+B(t))u(t), \quad t \in (s,T], \\ u(s) &= u_s \end{aligned} \tag{133}$$

defined in a complex and separable Hilbert space $\mathcal{H}$, with $A(t)+B(t)$ self-adjoint there. In this case, only partial results regarding the existence of dynamics are known, for example when the domain of $A(t)+B(t)$ is independent of time (see, for instance, [31], [34] and the references therein); but to the best of our knowledge a Trotter-Kato product formula for this is not available. For instance, in the case of (112) the corresponding quantum mechanical equation reads

$$\begin{aligned} i\frac{\partial u(x,t)}{\partial t} &= \left(-\frac{d^2}{dx^2} + \mathrm{s}(t)\delta_{x^*} + \varkappa + \varepsilon m(x,t)\right) u(x,t), \quad (x,t) \in \mathbb{R}\times(0,T]\,, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{134}$$

and under Hypotheses (S), (M$''$) there exist the unitary groups $\exp\left[-isA(t)\right]_{s\in\mathbb{R}}$ and $\exp\left[-isB_\varepsilon(t)\right]_{s\in\mathbb{R}}$ for every $t\in[0,T]$, where the $A(t)$'s, $B_\varepsilon(t)$'s are given by (116), (130), respectively. However, whether the strong limit

$$\lim_{n\to+\infty}\prod_{\gamma=n-1}^{0}\exp\left[-i\frac{t-s}{n}A\left(s+\frac{\gamma}{n}(t-s)\right)\right]\exp\left[-i\frac{t-s}{n}B_\varepsilon\left(s+\frac{\gamma}{n}(t-s)\right)\right]$$

exists in $\mathcal{L}(L^2(\mathbb{R},\mathbb{C}))$ *and* describes the true dynamics generated by (134) seems to be an open problem at this time. The same remark applies to other unitary evolution systems generated by Schrödinger equations in the presence of time-dependent singular perturbations of zero-range, such as those constructed in [32] and more recently in [8] and [12]. Away from the one-dimensional case, these constructions rest essentially on von Neumann's theory of self-adjoint extensions for symmetric operators.

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# References

[1] Acquistapace, P., Terreni, B., *A Unified Approach to Abstract Linear Nonautonomous Parabolic Equations,* Rendiconti del Seminario Matematico della Università di Padova **78** (1987) 47-107.

[2] Adams, R.A., Fournier, J.J.F., *Sobolev Spaces,* Pure and Applied Mathematics Series **140** (2003), Academic Press, New York.

[3] Albeverio, S., Gesztesy, F., Hoegh-Krohn R., Holden H., *Solvable Models in Quantum Mechanics,* Texts and Monographs in Physics (1988) Springer Verlag, New York.

[4] Babbitt, D., *The Wiener Integral and Perturbation Theory of the Schrödinger Operator,* Bulletin of the American Mathematical Society **70** (1964) 254-259.

[5] Cachia, V., Zagrebnov, V.A., *Operator-Norm Approximation of Semigroups by Quasi-Sectorial Contractions,* Journal of Functional Analysis **180** (2001) 176-194.

[6] Chernoff, P.R., *Note on Product Formulas for Operator Semigroups,* Journal of Functional Analysis **2** (1968) 238-242.

[7] Chernoff, P.R., *Semigroup Product Formulas and Addition of Unbounded Operators,* Bulletin of the American Mathematical Society **76** (1970) 395-398.

[8] Correggi, M., Dell'Antonio G.F., Figari R., Mantile, A., *Ionization for Three Dimensional Time-Dependent Point Interactions,* Communications in Mathematical Physics **257** (2005) 169-192.

[9] Da Prato, G., Zabczyk, J., *Stochastic Equations in Infinite Dimensions,* Encyclopedia of Mathematics and its Applications **44** (1992) Cambridge University Press, Cambridge.

[10] Davies, E.B., *One-Parameter Semigroups,* London Mathematical Society Monographs **15** (1980) Academic Press, New York.

[11] Dell'Antonio G.F., Figari R., Teta A., *A Limit Evolution Problem for Time-Dependent Point Interactions,* Journal of Functional Analysis **142** (1996) 249-275.

[12] Dell'Antonio G.F., Figari R., Teta A., *The Schrödinger Equation with Moving Point Interactions in Three Dimensions,* in: Stochastic Processes, Physics and Geometry: New Interplays I, CMS Conference Proceedings Series **28** (2000) 99-113, American Mathematical Society, Providence.

[13] Faris, W.G., *Product Formulas for Perturbations of Linear Propagators,* Journal of Functional Analysis **1** (1967) 93-108.

[14] Faris, W.G., *The Product Formula for Semigroups defined by Friedrichs Extensions,* Pacific Journal of Mathematics **22** (1967) 47-70.

[15] Faris, W.G., *Self-Adjoint Operators,* Lecture Notes in Mathematics **433** (1975) Springer Verlag, New York.

[16] Gulisashvili, A., van Casteren, J., *Non-Autonomous Kato Classes and Feynman-Kac Propagators* (2006) World Scientific, Singapore.

[17] Hardy, G.H., *Divergent Series* (1949) Clarendon Press, Oxford.

[18] Ichinose, T., Tamura, H., *Error Estimate in Operator Norm of Exponential Product Formulas for Propagators of Parabolic Evolution Equations,* Osaka Journal of Mathematics **35** (1998) 751-770.

[19] Johnson, G.W., Lapidus, M.L., *The Feynman Integral and Feynman's Operational Calculus,* Oxford Mathematical Monographs (2000) Oxford University Press, Oxford.

[20] Kato, T., *Abstract Evolution Equations of Parabolic Type in Banach and Hilbert Spaces,* Nagoya Mathematics Journal **5** (1961) 93-125.

[21] Kato, T., *Trotter's Product Formula for an Arbitrary Pair of Self-Adjoint Contraction Semigroups,* in: Topics in Functional Analysis (I. Gohberg and M. Kac, editors) (1978) 185-195, Academic Press, New York.

[22] Kato, T., *Perturbation Theory for Linear Operators,* Grundlehren der mathematischen Wissenschaften **132** (1984) Springer Verlag, New York.

[23] Kato, T., Tanabe, H., *On the Abstract Evolution Equation,* Osaka Journal of Mathematics **14** (1962) 107-133.

[24] Lions, J.L., *Équations Différentielles Opérationnelles et Problèmes aux Limites,* Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen **111** (1961) Springer Verlag, New York.

[25] Lions, J.L., *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires,* Études Mathématiques (1969) Dunod Gauthier-Villars, Paris.

[26] Neidhardt, H., Zagrebnov, V.A., *Trotter-Kato Product Formula and Operator-Norm Convergence,* Communications in Mathematical Physics **205** (1999) 129-159.

[27] Nelson, E., *Feynman Integrals and the Schrödinger Equation,* Journal of Mathematical Physics **5** (1964) 332-343.

[28] Nickel, G., *Evolution Semigroups and Product Formulas for Nonautonomous Cauchy Problems,* Mathematische Nachrichten **212** (2000) 101-116.

[29] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Applied Mathematical Sciences **44** (1983) Springer Verlag, New York.

[30] Räbiger, F., Rhandi, A., Schnaubelt, R., Voigt, J., *Non-Autonomous Miyadera Perturbations,* Differential and Integral Equations **13** (2000) 341-368.

[31] Reed, M., Simon, B., *Methods of Modern Mathematical Physics,* I, II (1975) Academic Press, London.

[32] Sayapova, M.R., Yafaev, D.R., *The Evolution Operator for Time-Dependent Potentials of Zero Radius,* Proceedings of the Steklov Institute of Mathematics **2** (1984) 173-180.

[33] Schnaubelt, R., *Semigroups for Nonautonomous Cauchy Problems,* in: One-Parameter Semigroups for Linear Evolution Equations (K.J. Engel and R. Nagel, editors) (2000) 477-496, Springer Verlag, New York.

[34] Simon, B., *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms,* Princeton Series in Physics (1971) Princeton University Press, Princeton.

[35] Tanabe, H., *Equations of Evolution,* Monographs and Studies in Mathematics **6** (1979) Pitman, London.

[36] Trotter, H., *On the Product of Semigroups of Operators,* Proceedings of the American Mathematical Society **10** (1959) 545-551.

[37] Vuillermot, P.-A., Wreszinski, W.F., Zagrebnov, V.A., *A Trotter-Kato Product Formula for a Class of Non-Autonomous Evolution Equations,* Trends in Nonlinear Analysis: in Honour of Professor V. Lakshmikantham, Nonlinear Analysis, Theory, Methods and Applications **69** (2008) 1067-1072.

[38] Yosida, K., *Functional Analysis,* Classics in Mathematics Series (1995) Springer Verlag, New York.

[39] Zagrebnov, V.A., *Quasi-Sectorial Contractions,* Journal of Functional Analysis **254** (2008) 2503-2511.